Journal of Statistical Physics, Vol. 74, Nos. 1/2, 1994

On the Correlation for Kac-like Models in the Convex Case

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Received May 5, 1993; final July 20, 1993

The aim of this paper is to study the behavior as m tends to ∞ of a family of measures $\exp[-\Phi^{(m)}(x)]\, dx^{(m)}$ on $\mathbb{R}^m$, where $\Phi^{(m)}$ is a potential on $\mathbb{R}^m$ which is a perturbation "in a suitable sense" of the harmonic potential $\sum_j x_j^2$.

KEY WORDS: Statistical mechanics correlation; thermodynamic limit magnetization; maximum principle.

1. INTRODUCTION

In recent publications we have given a new insight into old problems coming from statistical mechanics. This paper is in some sense the natural continuation of refs. 12 and 27. The semiclassical point of view will not play an important role (see, however, Section 8), but could be useful in order to give more precise results. We study here a family of measures on $\mathbb{R}^m$ parametrized by m of the form

$$d\mu^{(m)} = \exp[-\Phi^{(m)}(x)/h]\, dx^{(m)} \tag{1.1}$$

(where $dx^{(m)}$ is the Lebesgue measure on $\mathbb{R}^m$ and $\Phi^{(m)}$ is a suitable family of C^∞ potentials on $\mathbb{R}^m$) of the type introduced by Kac[18] and of different quantities attached to this measure. We consider first, for example, the thermodynamic limit, that is, the existence of the limit (and the speed of convergence) of

$$\lambda = -\lim_{m \to \infty} \ln \mu^{(m)}(\mathbb{R}^m)/m \tag{1.2}$$

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0022-4715/94/0100-0349$07.00/0

We shall denote by $a(m)$ the logarithm of the total measure:

$$a(m) = -\ln[\mu^m(\mathbb{R}^m)] = -\ln\left\{\int \exp[-\Phi^{(m)}(x)/h]\,dx\right\} \tag{1.3}$$

Most of the time in this article we shall take $h = 1$, but we refer to refs. 25–27, 15, 16, and 11 or 12 for studies of the semiclassical limit as h tends to $+0$. We shall restrict our study to the convex case and study particularly the speed of convergence in (1.2) and, if g is a function of k variables, we shall study the convergence of the mean value of g as m tends to ∞:

$$\langle g\rangle_\infty = \lim_{m\to\infty} \langle g\rangle_m \tag{1.4}$$

where

$$\langle g\rangle_{\Phi^m} = \langle g\rangle_m = \int g\,d\mu^{(m)}\Big/\int d\mu^{(m)} \tag{1.5}$$

As a particular but important case, we shall analyze the case when

$$g_{ij}(x) = x_i \cdot x_j \tag{1.6}$$

and will be interested in the behavior as $|i-j| \to \infty$ of the correlation

$$\mathrm{Cor}(i, j) = \langle g_{ij}\rangle_\infty - \langle x_i\rangle_\infty \langle x_j\rangle_\infty \tag{1.7}$$

We shall prove in the convex case (in connection with results of Sokal[30] mentioned in the book of Ellis[7] an exponential decay with respect to $|i-j|$. Let us now describe the results and the ideas. We first give a simple criterion for the existence of the thermodynamic limit:

Proposition 1.1. Let C be a positive constant and let us consider the following family of potentials indexed by $m \in \mathbb{N}$:

$$\Phi^{(m)}(x) = \frac{x^2}{2} + \Psi^{(m)}(x)$$

satisfying the following properties:

$$\Psi^{(m)}(0) = 0 \tag{1.8}$$

$$|\nabla\Psi^{(m)}|_{l^\infty} \leqslant C \tag{1.9}$$

$$|\nabla(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))|_{l^1} \leqslant C \tag{1.10}$$

$$|\Delta(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))| \leqslant C \tag{1.11}$$

Then the thermodynamic limit exists and the speed of the convergence to the limit is controlled in D/m:

$$|\lambda + [\ln \mu^{(m)}(\mathbb{R}^m)/m]| \leqslant D/m \tag{1.12}$$

where D is explicitly computable from C.

We have used in the proposition the notation

$$(\Psi^{(m)} \oplus \Psi^{(n)})(x) = \Psi^{(m)}(x') + \Psi^{(n)}(x''), \qquad \forall x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^n$$

Corollary 1.2. Let W be a C^2-function on $\mathbb{R}^2$ such that ∇W and ΔW are bounded. Then the assumptions of Proposition 1.1 are satisfied for

$$\Psi^{(m)}(x) = \sum_{i=1}^{m} W(x_i, x_{i+1}) \tag{1.13}$$

with the convention that $m+1=1$.

This corollary was obtained in ref. 11 or ref. 12, but similar results are of course older (see, for example, ref. 20).

Example 1.3.

$$W(u, v) = -\ln \cosh[\sqrt{\nu}\,(u+v)]$$

This example was one of the motivating examples for our recent studies in refs. 25–27, 15, 16, 11, and 12 and comes from Kac.[18] The corresponding family of potentials is convex for $\nu \leqslant 1/4$.

In the proof of Proposition 1.1 we introduce the intermediate family (parametrized by $s \in [0, 1]$)

$$\Phi^{(m,n)}(x, s) = ((1-s)\Phi^{(m+n)} + s(\Phi^{(m)} \oplus \Phi^{(n)}))$$

and the idea is to prove the boundedness of the logarithmic derivative:

$$-\frac{\partial}{\partial s} \ln \int_{\mathbb{R}^{m+n}} \exp -\Phi^{(m,n)}(x, s)\, dx = \langle \partial_s \Phi^{(m,n)}(\cdot, s) \rangle_{\Phi^{(m,n)}}$$

which is done by an integration by parts based on the trivial identity

$$x_j \exp -\tfrac{1}{2}x^2 = -\partial_{x_j} \exp -\tfrac{1}{2}x^2$$

We next discuss the more sophisticated methods leading to exponential estimates. Our basic idea is to investigate the mean value $\langle c \rangle_\Phi$ of a temperate function c by first solving

$$c = (-\nabla\Phi \cdot \nabla + \Delta)u + b \tag{1.14}$$

where b is a constant, and notice that by integration by parts we have

$$b = \langle c \rangle_\Phi$$

Similarly for the correlation

$$\mathrm{Cor}(c, d) = \langle (c - \langle c \rangle)(d - \langle d \rangle) \rangle_\Phi$$

of two functions c, d, if $u = u_c$ solves (1.14), then we get

$$\mathrm{Cor}(c, d) = \langle \nabla d \cdot \nabla u_c \rangle$$

and ∇u is a solution of

$$\nabla c = (-\nabla\Phi \cdot \nabla + \Delta)\nabla u + \mathrm{Hess}\,\Phi\,\nabla u \tag{1.15}$$

When Φ is strictly convex (with additional assumptions) it turns out that we can get good pointwise estimates of weighted norms of ∇u in terms of corresponding norms of ∇c. Assume now that

$$\mathrm{Hess}\,\Phi(x) \geqslant \delta_0 > 0 \tag{1.16}$$

$$\|\mathrm{Hess}\,\Phi(x)\|_{\mathscr{L}(l_\rho^\infty)} \leqslant C \tag{1.17}$$

(where all the constants are independent of the dimension m if nothing else is indicated) for all ρ on $\mathbb{Z}/m\mathbb{Z}$ satisfying

$$e^{-\kappa} \leqslant \frac{\rho(\nu+1)}{\rho(\nu)} \leqslant e^{\kappa}, \qquad \nu \in \mathbb{Z}/(m+n)\mathbb{Z} \tag{1.18}$$

for $\kappa > 0$, and

$$\|I - \mathrm{Hess}\,\Phi\|_{\mathscr{L}(l_\rho^\infty)} \leqslant \delta < 1 \tag{1.19}$$

Here l_ρ^∞ is the weighted l^∞-space defined by the norm

$$|x|_{l_\rho^\infty} = |\rho x|_\infty = \max_\nu |\rho(\nu) x_\nu|$$

Then we have the following theorem:

Theorem 1.4. If

$$\mathrm{Cor}^{(m)}(i, j) = \langle x_i x_j \rangle_m - \langle x_i \rangle_m \langle x_j \rangle_m$$

then under the assumptions (1.16)–(1.19) we have

$$|\mathrm{Cor}^{(m)}(i, j)| \leqslant C_\varepsilon \exp -[(\kappa - \varepsilon)\,\mathrm{dist}_{\mathbb{Z}/m\mathbb{Z}}(i, j)] \tag{1.20}$$

for all m, $\varepsilon > 0$ and all pairs (i, j).

We shall give in Section 7 a slightly more general result implying this one. In particular we get immediately the following corollary.

Theorem 1.5. Let W be a C^3 function on $\mathbb{R}^2$ such that $\nabla^p W$ is bounded (for $p=1, 2, 3$). Then for

$$\Psi^{(m)}(x) = \beta \sum_{i=1}^{m} W(x_i, x_{i+1})$$

the assumptions of Theorem 1.4 are satisfied for $|\beta|$ small enough.

The idea of the proof is to use the maximum principle to estimate $\|\nabla u\|_{l^\infty_\rho}$ with $\rho(j) = \exp[(\kappa - \varepsilon)\, d(j, k)]$, where u solves (1.14) with $c = x_k$ and $d(j, k) = \mathrm{dist}_{\mathbb{Z}/m\mathbb{Z}}(j, k)$.

Similar estimates can be given for more general correlations.

A similar method can be used to prove exponential convergence to the thermodynamic limit. Consider the family $\Phi^{(m)}$ as in Proposition 1.1. Let us assume

$$\nabla \Psi^{(m)}(0) = 0 \tag{1.21}$$

and that, for some C, $\kappa > 0$, and $0 < \delta < 1$, the following properties are fullfilled:

$$\Psi^{(m)}(x_1, x_2,\dots, x_m) = \Psi^{(m)}(x_2, x_3,\dots, x_m, x_1) \tag{1.22}$$

Similarly to the conditions of Theorem 1.4, we impose the following conditions that the intermediate family $\Phi^{(m,n)}(x, s)$ satisfies for any $s \in [0, 1]$, m and n in $\mathbb{N}$ $(m \geqslant 1;\ n \geqslant 1)$, $x = (x', x'') \in \mathbb{R}^{m+n}$:

$$(\mathrm{Hess}_x\, \Phi^{(m,n)}(x, s)) \geqslant \delta_0 > 0 \tag{1.23}$$

$$\|\mathrm{Hess}\, \Phi^{(m,n)}(x, s)\|_{\mathscr{L}(B_1)} \leqslant C \tag{1.24}$$

with

$$B_1 = (\mathbb{R}^{(m+n)}, \|\cdot\|_{l^\infty}) \tag{1.25}$$

We also introduce the family of normed spaces

$$B = l^\infty_{\rho_{m,n}} \tag{1.26}$$

(with underlying vector space $\mathbb{R}^{(m+n)}$) and where $\rho_{m,n}$ is in the family of weights defined on $\mathbb{Z}/(m+n)\mathbb{Z}$ satisfying (1.18) and

$$\rho_{m,n}(1) = \rho_{m,n}(m) = 1 \tag{1.27}$$

We shall denote by $\mathscr{B}_2$ the family of these B's associated to some κ and we assume

$$\|\nabla^2(\Phi^{(m,n)}(x,s))\|_{\mathscr{L}(B,B)} \leqslant C \qquad \text{for any} \quad B \in \mathscr{B}_2 \tag{1.28}$$

and more precisely

$$\|(I - \text{Hess}_x\, \Phi^{(m,n)})(x,s)\|_{\mathscr{L}(B)} \leqslant \delta < 1 \tag{1.29}$$

Similarly to the assumptions given in Proposition 1.1, we impose also the following conditions on $(\partial_s \Phi^{(m,n)})(x,s) = (\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))$:

$$|\nabla(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(x)|_B \leqslant C \qquad \text{for any} \quad B \in \mathscr{B}_2 \tag{1.30}$$

[this is a refinement of (1.10)]

$$\|\nabla^2(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(x)\|_{\mathscr{L}(B_1,B)} \leqslant C \qquad \text{for any} \quad B \in \mathscr{B}_2 \tag{1.31}$$

[this is a refinement of (1.11)]. Under these assumptions and other technical conditions of the same type for derivatives of order 3 and 4 which will be given in Section 6.4 [(6.25)–(6.27)] we get the following theorem:

Theorem 1.6. Under these assumptions we have

$$|a(m)/m - \lim_{m \to \infty} a(m)/m| \leqslant E \exp -\kappa m/4 \tag{1.32}$$

for some constant E (independent of m).

This is the analog of Theorem 3.1 in ref. 27. The assumption

$$\nabla \Psi^{(m)}(0) = 0$$

corresponds to the fact that the unique critical point is supposed to be 0. This is automatically satisfied if Ψ is an even function. In some cases (see the magnetization problem) this condition is not satisfied, but we can come back to this case by a translation argument. This problem will be discussed in more detail in Section 6 (Propositions 6.2 and 6.3).

We shall see in Section 14 how this result can be applied in the case where $\Phi^{(m)} = -\ln(u^{(m)})$, where $u^{(m)}$ is the first normalized eigenfunction of a Schrödinger operator where the potential $V^{(m)}$ satisfies essentially the same assumptions as in Theorem 1.6.

Outline of the Proof. The theorem will be proved in Section 6.4, but we here outline a possible slight variation in the special case where

$$\Phi^{(m)}(x) = \tfrac{1}{2}x^2 + \sum W(x_i, x_{i+1})$$

in order to emphasize the role of the correlations. Consider, for instance, the two-parameter family

$$\tilde{\Phi}^{(2m)}(x, t, s) = \tfrac{1}{2}x^2 + \sum_{i=1}^{2m-1} W(x_i, x_{i+1}) + tW(x_{2m}, x_1) + (s-1)\, W(x_k, x_{k+1})$$

where $\operatorname{dist}(k, 1) \geqslant m/3$. Then we shall use the classical formula

$$\partial_s \partial_t \ln \int_{\mathbb{R}^{2m}} \exp -\tilde{\Phi}^{(2m)}\, dx = \langle \partial_s \partial_t \tilde{\Phi}^{(2m)} \rangle - \operatorname{Cor}(\partial_t \tilde{\Phi}^{(2m)}, \partial_s \tilde{\Phi}^{(2m)})$$

Here the first term in the r.h.s. vanishes and we can apply a result of the type given in Theorem 1.4 in order to prove that the correlation term is $\mathcal{O}(1) \exp -\kappa m$ for some $\kappa > 0$. Up to an exponentially small term this means that $\ln \int \exp -\tilde{\Phi}\, dx$ is of the form $f(t) + f(s)$. In other words, adding an interaction between k and $k+1$ and adding an interaction between $2m$ and 1 are essentially independent. Putting four modifications at suitable places, we then see that

$$\ln \int \exp -\Phi^{(2m)}\, dx^{(2m)} = 2 \ln \int \exp -\Phi^{(m)}\, dx^{(m)} + \mathcal{O}(1) \exp -\kappa m$$

for some κ. ∎

We shall also study the family of measures on cylindrical functions and obtain:

Proposition 1.7. Let c be a C^∞ function defined on $\mathbb{R}^k$ s.t. $\nabla c \in C_b^\infty(\mathbb{R}^k)$ that we identify with a function on $\mathbb{R}^m$ by

$$c^{(m)}(x) = c(x_1, x_2, \ldots, x_k)$$

Then, under the assumptions of Theorem 1.6, the limit as m tends to ∞ of $\langle c^{(m)} \rangle_m$ exists and the convergence is exponentially fast as in (1.32).

Another application is the study of the magnetization in the case of the following family of potentials:

$$\Phi^{(m)}(x, B) = \tfrac{1}{2} \sum_l (x_l - B)^2 + \Psi^{(m)}(x) \tag{1.33}$$

Then the magnetization is defined by

$$\mathcal{M}(m, B) = (1/m)(\partial a/\partial B)(m, B) \tag{1.34}$$

or $\mathcal{M}(m, B) = \langle x_1 - B \rangle_m$.

Theorem 1.8. Under the same assumptions as in Theorem 1.5, the magnetization $\mathcal{M}(m, B)$ is convergent (exponentially rapidly) to a continuous limit. In particular, if $\Psi^{(m)}$ is even, then the limit as B tends to 0 is zero.

The last statement in the theorem can be interpreted as a statement of absence of magnetization in the convex case, which of course is not surprising.

This paper is organized in three parts. Part I is devoted to the existence of the various limits mentioned above. Section 2 is technical and devoted to the existence and uniqueness of a solution for the "basic" equation [cf. (1.15)]

$$\nabla g = (\nabla\Phi\nabla - \Delta)v + \text{Hess}\,\Phi v$$

with $v = f^{(1)} = \nabla f$, where Φ is defined above and g is a suitable C^∞ function which is slowly increasing. Section 3 is devoted to precise estimates using the maximum principle in the spirit of refs. 24 and 27. Here the convexity of the family plays an important role. These estimates are first obtained for $\Psi^{(m)}$ and g's with compact support. Section 4 will analyze weighted estimates for higher Hessians. The argument of cutoff permitting the elimination of this restriction is explained in Section 5. Section 6 is devoted to the study of the thermodynamic limit and Section 7 to the study of the "limit" measure and the proof of Proposition 1.7.

Part II is devoted to more precise estimates for the correlations. In Section 8, we make a preliminary analysis of the problem using a semiclassical heuristical approach. In Section 9, we establish weighted $l^1 \to l^\infty$ estimates for inverse matrices. In Section 10, we come back to improvements of estimates given in Section 3. Section 11 treats one basic example which was suggested by statistical mechanics and motivates the title of our paper. Section 12 is devoted to a new approach of the sign of the correlation in connection with the celebrated FKG inequalities. Section 13 is devoted to the study of the higher-order correlations.

Part III is devoted to the Schrödinger equation and we analyze how the results of Part II give apparently new results for the correlation or the magnetization. This will be the object of Section 14.

PART I. ON THE EXISTENCE OF DIFFERENT THERMODYNAMIC LIMITS AS THE DIMENSION m TENDS TO ∞

2. EXISTENCE OF SOLUTIONS FOR A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

Let us consider in this section the problem of solving in suitable spaces the following problem:

$$w = (-\Delta + \nabla\Phi\cdot\nabla)v + (\text{Hess}\,\Phi)v \tag{2.1}$$

Here w is a C^∞ vector field on $\mathbb{R}^m$ and the unknown is the vector field v. The aim of this section is to prove the existence of a unique "temperate" solution of the equation if w is temperate. More precisely, we shall prove the following proposition:

Proposition 2.1. If $\Phi(x) = x^2/2 + \Psi(x)$ with $\Psi(x)$ satisfying

$$|(\partial_x^\alpha \Psi)| \leqslant C_\alpha \tag{2.2}$$

for $|\alpha| \geqslant 1$ and

$$\text{Hess } \Phi \geqslant \rho > 0 \tag{2.3}$$

then for any C^∞ vector field w satisfying

$$|(\partial_x^\alpha w)| \leqslant C_\alpha (1 + |x|)^{q_\alpha} \tag{2.4}$$

for every $\alpha \in \mathbb{N}^m$ with some C_α and some q_α, there exists a unique C^∞ vector v solution of (2.1) such that

$$\exp[-\Phi(x)/2] v \in B^1(\mathbb{R}^m; \mathbb{R}^m) \tag{2.5}$$

where, for $k, p, m \in \mathbb{N}$,

$$B^k(\mathbb{R}^m; \mathbb{R}^p) = \{u \in L^2(\mathbb{R}^m);\ x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}^m; \mathbb{R}^p) \text{ for } |\alpha| + |\beta| \leqslant k\} \tag{2.6}$$

Proof of Proposition 2.1. We just come back to an easier situation by making the following change of function:

$$u = \exp(-\Phi/2) v; \qquad q = \exp(-\Phi/2) w \tag{2.7}$$

The system (2.1) is transformed in the following system for (u, q):

$$(-\Delta + |\nabla\Phi|^2/4 - \Delta\Phi/2) u + \text{Hess } \Phi u = q \tag{2.8}$$

Our assumptions imply in particular that q is in $L^2(\mathbb{R}^m; \mathbb{R}^m)$ and we can find u as the unique solution of the associated solution of the variational problem on the Sobolev space $B^1(\mathbb{R}^m; \mathbb{R}^m)$. We define as usual by duality the spaces $B^k(\mathbb{R}^m; \mathbb{R}^p)$ for $k \in \mathbb{Z}$; $k < 0$ and using the positivity of the operator

$$(-\Delta + |\nabla\Phi|^2/4 - \Delta\Phi/2) = (\partial_x + \nabla\Phi/2)^* (\partial_x + \nabla\Phi/2)$$

and the strict positivity of the Hessian given by assumption (2.3), we see that the solution u is unique in $B^1(\mathbb{R}^m; \mathbb{R}^m)$ for any q in $B^{-1}(\mathbb{R}^m; \mathbb{R}^m)$. Then classical results (see, for example, ref. 19) on the global regularity of the operator give that if q is in $B^\infty(\mathbb{R}^m; \mathbb{R}^m)$, then u is also in $B^\infty(\mathbb{R}^m; \mathbb{R}^m)$. One way here is to use the fact that the inverse is a pseudodifferential operator which is continuous from B^k into B^{k+2} for any k in $\mathbb{N}$. Another way is to use the method of the differential quotients. The results announced in Proposition 2.1 are then easy to obtain. ∎

Remark 2.2. It is relatively classical to prove an analog of Proposition 2.1 with dependence with respect to a parameter s. If we consider a family $\Phi(x, s) = x^2/2 + \Psi(x, s)$ and $w(x, s)$ with (1) $\Psi(x, s)$ satisfying (2.2), (2.3) uniformly with respect to s, (2) $\partial_s \Psi(x, s)$ satisfying (2.2) uniformly with respect to s, and (3) w and $\partial_s w(x, s)$ satisfying (2.4) uniformly with respect to s, then u and $\partial_s u$ are in $B^\infty(\mathbb{R}^m; \mathbb{R}^m)$ uniformly with respect to s in $[0, 1]$. In particular, $(\partial_x^\alpha v)$ and $(\partial_s \partial_x^\alpha v)$ are continuous functions of (x, s) with values in $\mathbb{R}^m$.

We observe here that the result is rather weak, but will be sufficient for us and in this section we shall give more precise results under the assumption that Ψ and w have compact support and under weaker assumptions in Section 5 as a consequence of the maximum principle estimates obtained in Section 4. This is the object of the next proposition:

Proposition 2.3. Under the assumption (2.3), if w and Ψ are compactly supported, then the solution v constructed in Proposition 2.1 satisfies the following property:

$$|\partial_x^\alpha v(x)| \leqslant C_\alpha \langle x \rangle^{-1/2-|\alpha|} \tag{2.9}$$

Proof of Proposition 2.3. The proof of the preceding proposition gives us only an exponential control. We follow here the proof given in a similar context in ref. 27. Actually, what we need is to give an asymptotic expansion of the solution at ∞. For this we choose a large ball B containing the supports of u and q and with a radius greater than $\sqrt{m}$. The solution u we have found in the preceding proof is an L^2 solution of

$$u/\partial B = r; \qquad (-\Delta + |x|^2/4 - m/2 + 1)u = 0, \qquad x \in \complement B \tag{2.10}$$

where r is the C^∞-vector field on ∂B defined as the trace of u on ∂B. We observe that this system is a diagonal system and we can now work components by components and the situation is reduced to the scalar case which is essentially treated in ref. 27. Let us establish the following proposition (for the scalar case):

Proposition 2.4. The L^2 solution u of

$$u/\partial B = r; \qquad (-\Delta + |x|^2/4 - m/2 + 1)u = 0, \qquad x \in \complement B \tag{2.11}$$

satisfies

$$u(x) = \exp(-|x|^2/2)\,|x|^{-1/2}\,h(x) \tag{2.12}$$

where $h(x)$ is C^∞ and satisfies

$$D_x^\beta h(x) = O(|x|^{-|\beta|}), \qquad \forall \beta \in \mathbb{N}^m \tag{2.13}$$

Proof of Proposition 2.4. As in ref. 27, we observe by the maximum principle that the operator K which associates to r the solution u of (2.11) is monotone. In particular, if we introduce

$$u_0 = K1 \tag{2.14}$$

we get the positivity of u_0.

We write the general solution u in the form

$$u(x) = u_0(x)\,j(x)$$

and we get from the maximum principle that j is bounded. We observe also that using techniques of differential equations, we can get a complete asymptotics of u_0 and of its derivatives at ∞. Observing that $j(x)$ plays the role of $\exp k$ in Section 1 in ref. 27, we see that the proof given in this paper between (1.10) and (1.18) is unchanged and gives the result. ∎

Using this proposition, for each of the components of the vector field u, we get immediately Proposition 2.3. ∎

Remark 2.5. *Dependence with respect to a parameter s.* We observe here that, under the conditions of Proposition 2.4 (assumed to be uniform with respect to s) and the conditions added in Remark 2.2, we obtain that (2.9) is also true with v replaced by $\partial_s v$. The only point to remark is that the restriction to ∂B of u is regular in (x, s) according to Remark 2.2.

3. ESTIMATES BY THE MAXIMUM PRINCIPLE (PRELIMINARIES)

3.1. Introduction of the Basic Equation

In this section (and the next one), we forget the explicit reference to m, but let us emphasize that all the dependence on this parameter is controlled very explicitly once the norms are chosen. In particular, we never use the general result of equivalence of the different norms on a finite-dimensional normed space, because the constants appearing in this equivalence depend in general on the dimension. We shall work with a function Φ of the form $x^2/2 + \Psi$ with Ψ compactly supported, and we shall use in this section the results of existence given in the preceding section. In particular, we shall always have the property, when we have to apply the maximum principle, that the functions tend to 0 at ∞. Our problem is here to deduce universal estimates relating the solution of the equation to the assumptions. These universal estimates will allow us in the next section to justify estimates in the general situation by a control of the cutoff argument. If g is a C^∞ function with compact support, we shall meet in Section 6 the problem of finding a contant b and a vector field v such that

$$g = b + v \cdot \nabla_x \Phi - \operatorname{div}_x v \qquad \text{for} \quad x \in \mathbb{R}^m \tag{3.1}$$

Then we shall see that it is natural to derive with respect to x and that we shall find this constant after using, for example, that

$$g(0) = 0, \qquad \nabla_x \Phi(0) = 0$$

We then obtain the equation (where v is a vector field on $\mathbb{R}^m$)

$$\partial_j g = \partial_j (v_l \partial_l \Phi - \partial_l v_l)$$

(with the usual conventions of summation) and

$$\partial_j g = (\partial_j v_l) \cdot (\partial_l \Phi + v_l \cdot \partial^2_{lj} \phi - \partial^2_{l,j} v_l)$$

If we look for a v as a gradient, we observe that

$$\partial^2_{l,j} v_l = \partial^2_{l,l} v_j$$

and that

$$\partial_j v_l = \partial_l v_j$$

and we obtain the equation

$$w = (-\Delta + \nabla \Phi \cdot \nabla) v + (\text{Hess } \Phi) v \tag{3.2}$$

with

$$w = \nabla g \tag{3.3}$$

Equation (3.2) is the equation that we discussed in Section 2 and we call it the "basic" equation.

3.2. On the Uniqueness and the Existence of "Gradient Solutions"

In order to be coherent, we recall that we have to verify first that if v is a solution of (3.2) and if w is a gradient, then v is a gradient, that is, there exists a function f such that $v = \nabla f$. This will be obtained also as an application of the maximum principle:

Proposition 3.1. Let v a C^3-solution of (3.2) s.t. all the derivatives of v of order $\leqslant 2$ tend to 0 as $|x| \to \infty$. Let us assume that

$$(\text{Hess}\, \Phi)(x) \geqslant I/C \qquad (\text{with } C > 0) \tag{3.4}$$

If w satisfies $\partial_l w_i = \partial_i w_l$, then the same is true for v.

Proof of Proposition 3.1. As a preliminary remark, we explain how to recover the uniqueness of v through the maximum principle. For the moment, the choice of the norm on $\mathbb{R}^m$ is not decisive and we choose the l^2 norm. Because v tends to 0 at ∞, we can consider a point x_0 at which $\|v(x)\|$ is maximal. Taking the scalar product in (3.2) with $v(x_0)$, we get the equation

$$\langle w(x) \mid v(x_0) \rangle = (-\Delta + \nabla\Phi \cdot \nabla) \langle v(x) \mid v(x_0) \rangle + \langle (\text{Hess}\, \Phi)\, v(x) \mid v(x_0) \rangle \tag{3.5}$$

Taking $x = x_0$, we obtain

$$\langle w(x_0) \mid v(x_0) \rangle \geqslant \langle (\text{Hess}\, \Phi)(x_0)\, v(x_0) \mid v(x_0) \rangle \tag{3.6}$$

where we have used that $\langle v(x) \mid v(x_0) \rangle$ takes its maximum at x_0. We use now the strict convexity assumption on Φ given in (3.4):

$$\sup_x \|v(x)\| \leqslant C \sup_x \|w(y)\| \tag{3.7}$$

and in particular the uniqueness of v.

In order to prove the proposition, we derive once again the equation with respect to x_i, and we get

$$\partial_i w_j = \partial_l \Phi \cdot \partial_{li}^2 v_j + v_l \cdot \partial_{l,j,i}^3 \Phi - \partial_{lli}^3 v_j + \partial_{l,i}^2 \Phi \cdot \partial_l v_j + \partial_i v_l \cdot \partial_{l,j}^2 \Phi \tag{3.8}$$

Let us introduce

$$v_{ij}^{\text{as}} = (\partial_i v_j - \partial_j v_i)/2; \qquad w_{ij}^{\text{as}} = (\partial_i w_j - \partial_j w_i)/2$$

the antisymmetric parts of the matrices $(\partial_i v_j)$ and $(\partial_i w_j)$. We then deduce from (3.8) the following system:

$$w_{ij}^{\text{as}} = (-\partial_{ll}^2 + \partial_l \Phi \cdot \partial_l) v_{ij}^{\text{as}} + (\partial_{il} \Phi) v_{l,j}^{\text{as}} + v_{il}^{\text{as}} \cdot \partial_{lj} \Phi \tag{3.9}$$

This new equation can be seen as an equation between antisymmetric matrices, which we write

$$W^{as} = (-\Delta + \nabla\Phi \cdot \nabla) V^{as} + (\text{Hess } \Phi) \circ V^{as} + V^{as} \circ (\text{Hess } \Phi) \tag{3.10}$$

We use again the maximum principle and take the Hilbert–Schmidt norm on the antisymmetric matrices. We recall that this is the norm attached to the following scalar product (we work with real matrices):

$$(V_1, V_2) \to -\text{tr}(V_1 \circ V_2)$$

As in the proof of (3.7), we get at a point x_1 where $\|V_x\|$ is maximal the inequality

$$\langle W^{as}(x) \mid V^{as}(x_1) \rangle \geqslant -2 \,\text{tr}((\text{Hess } \Phi)(x_1) \circ V(x_1) \circ V(x_1))$$

and finally

$$\sup_x \|V^{as}(x)\| \leqslant 2C \sup_x \|W^{as}(x)\| \quad \blacksquare \tag{3.11}$$

4. EXPONENTIALLY WEIGHTED ESTIMATES FOR HIGHER-ORDER HESSIANS

All estimates are uniform with respect to m, unless otherwise specified. Let $\mathscr{R} = \mathscr{R}^m$ be a set of weights ρ: $\{1,..., m\} \to]0, +\infty[$ satisfying

$$\rho, \mu \in \mathscr{R} \Rightarrow \rho^t \mu^{1-t}, \qquad \frac{1}{\rho} \in \mathscr{R}, \qquad 0 \leqslant t \leqslant 1, \qquad \lambda\rho \in \mathscr{R}, \qquad \forall \lambda \in]0, +\infty] \tag{4.1}$$

Let

$$\mathscr{R}_k = \{(\rho_1,..., \rho_k) \in \mathscr{R}^k; \rho_{\mathscr{J}} \in \mathscr{R}, \mathscr{J} \subset \{1,..., k\}\} \tag{4.2}$$

Here $\rho_{\mathscr{J}}(v) = \prod_{j \in \mathscr{J}} \rho_j(v)$, $\rho_\varnothing \equiv 1$.

We shall also need some extra parameters $s_1,..., s_p$, belonging to some compact set $\mathscr{S}$ in $\mathbb{R}^p$, and we associate a weight $\tilde{\rho}_j \in \mathscr{R}$ to each s_j. Following the definition introduced in ref. 29, a function $u \in C^\infty(\mathbb{R}^m; \mathbb{R})$ is called 0-*standard* (in the parameter-dependent sense) if (uniformly in L, x, ρ, $\tilde{\rho}$, s_j) for $k \geqslant 1$

$$\langle \nabla^k \partial_s^L u, t_1 \otimes \cdots \otimes t_k \rangle = \mathcal{O}_{k, \# L}(1) \prod_{j=1}^k |t_j|_{\rho_j, \rho_j} \qquad \left(\partial_s^L := \prod_{l \in L} \partial_{s_l}\right) \tag{4.3}$$

when

$$1=\sum \frac{1}{p_j}, \qquad 1 \leqslant \prod_{l \in L} \tilde{\rho}_l \cdot \prod_1^k \rho_j \tag{4.4}$$

and

$$((\tilde{\rho})_L, \rho_1, \ldots, \rho_k) \in \mathscr{R}_{(\# L+k)}, \qquad \tilde{\rho}_l \leqslant \hat{\rho}_l$$

Here we recall also that it is convenient to write

$$t_1(\partial_x) \cdots t_k(\partial_x) u = \langle \nabla^k u(x), t_1 \otimes \cdots \otimes t_k \rangle$$

and that Δ, ∇, and Hess refer to derivations with respect to x.

Let us now come back to our basic equation:

$$g = (\nabla \Phi \cdot \partial_x - h\Delta) f - c, \qquad c = \text{const} \tag{4.5}$$

Under suitable assumptions on Φ, we shall show that f is 0-standard when g is. The assumptions are:

Φ satisfies (4.3) for L, x, ρ, $\tilde{\rho}$, s_j, k, but here with the additional condition that

$$k \geqslant 2 \qquad \text{for} \quad L = \varnothing \tag{4.6}$$

and

$$\exists \delta \in [0, 1[\qquad \text{s.t.} \quad \|\text{Hess}\, \Phi(x) - I\|_{\mathscr{L}(l^p_\rho)} \leqslant \delta \tag{4.7}$$

$$\text{for} \quad \rho \in \mathscr{R}, \qquad 1 \leqslant p \leqslant \infty, \qquad x \in \mathbb{R}^m,\ s \in \mathscr{S}$$

We have

$$\begin{aligned} &t_1(\partial_x) \cdots t_k(\partial_x)\, \partial_s^L(\nabla \Phi \cdot \nabla f) \\ &\quad = \nabla \Phi \cdot \partial_x (t_1(\partial_x) \cdots t_k(\partial_x)\, \partial_s^L f) \\ &\qquad + \sum_{j=1}^k \langle t_j(\partial_x) \cdot \nabla \Phi, t_1(\partial_x) \cdots \widehat{t_j(\partial_x)} \cdots t_k(\partial_x)\, \partial_s^L \nabla f \rangle \\ &\qquad + \sum_{\substack{\mathscr{J} \cup \mathscr{K} = \{1, \ldots, k\},\ \mathscr{J} \cap \mathscr{K} = \varnothing \\ \tilde{\mathscr{J}} \cup \tilde{\mathscr{K}} = L,\ \tilde{\mathscr{J}} \cap \tilde{\mathscr{K}} = \varnothing \\ \# \tilde{\mathscr{K}} = L \Rightarrow \# \mathscr{K} \leqslant k-2}} \langle t_{\mathscr{J}}(\partial_x)\, \partial_s^{\tilde{\mathscr{J}}} \nabla \Phi, t_{\mathscr{K}}(\partial_x)\, \partial_s^{\tilde{\mathscr{K}}} \nabla f \rangle \end{aligned} \tag{4.8}$$

Here we notice for later use that the sum of the first two terms of the r.h.s. of (4.8) can also be written as

$$\begin{aligned} &\nabla \Phi \cdot \partial_x (t_1(\partial_x) \cdots t_k(\partial_x)\, \partial_s^L f) + k \langle \nabla^k \partial_s^L f, t_1 \otimes \cdots \otimes t_k \rangle \\ &\quad + \sum_{j=1}^k \langle \nabla^k \partial_s^L f, t_1 \otimes \cdots \otimes (\text{Hess}\, \Phi - I)\, t_j \otimes \cdots \otimes t_k \rangle \end{aligned} \tag{4.9}$$

Proposition 4.1. Assume that g is 0-standard and that (4.5) has a smooth solution f such that $\nabla^k f(x) \to 0$ as $|x| \to \infty$ for every $k \geqslant 1$. Then f is 0-standard.

Proof. Differentiating (4.5) and using (4.8) and (4.9), we get

$$\begin{aligned}
&\langle \nabla^k \partial_s^L g, t_1 \otimes \cdots \otimes t_k \rangle \\
&\quad = (\nabla \Phi \cdot \partial_x - h\Delta)(\langle \nabla^k \partial_s^L f, t_1 \otimes \cdots \otimes t_k \rangle) \\
&\qquad + k \langle \nabla^k \partial_s^L u, t_1 \otimes \cdots \otimes t_k \rangle \\
&\qquad + \sum_{j=1}^{k} \langle \nabla^k \partial_s^L f, t_1 \otimes \cdots \otimes (\text{Hess}\, \Phi - I)\, t_j \otimes \cdots \otimes t_k \rangle \\
&\qquad + \sum_{\substack{\mathcal{J} \cup \mathcal{K} = \{1,\dots,k\},\, \mathcal{J} \cap \mathcal{K} = \varnothing \\ \tilde{\mathcal{J}} \cup \tilde{\mathcal{K}} = L,\, \tilde{\mathcal{J}} \cap \tilde{\mathcal{K}} = \varnothing \\ \# \tilde{\mathcal{K}} = L \Rightarrow \# \mathcal{K} \leqslant k-2}} \langle t_{\mathcal{J}}(\partial_x)\, \partial_s^{\tilde{\mathcal{J}}}\, \nabla \Phi, t_{\mathcal{K}}(\partial_x)\, \partial_s^{\tilde{\mathcal{K}}}\, \nabla f \rangle \qquad (4.10)
\end{aligned}$$

We proceed by induction over L, adding an element stepwise, and for each L, we make an induction over $k = 1, 2,\dots$:

Step 1. Let $L = \varnothing$, $k = 1$. We observe that we have $p_1 = 1$. Then (4.10) reduces to

$$\langle \nabla g, t_1 \rangle = (\nabla \Phi \cdot \partial_x - h\Delta)(\langle \nabla f, t_1 \rangle) + \langle \nabla f, t_1 \rangle + \langle \nabla f, (\text{Hess}\, \Phi - I) t_1 \rangle \qquad (4.11)$$

and we are supposed to estimate $|\nabla f|_\infty$.

Let $m_0 = \sup_x |\nabla f(x)|_\infty$. If $m_0 = 0$, there is nothing to prove. If $m_0 \neq 0$, let x_0 be a point where $m_0 = |\nabla f(x_0)|$ and let $t_1 \in l^1$ be a normalized vector with $m_0 = (\nabla f(x_0), t_1 \rangle$. Then $(\nabla \Phi \cdot \partial_x - h\Delta)(\langle \nabla f, t_1 \rangle) \geqslant 0$ at x_0, so (4.11) and (4.7) give

$$(1-\delta) m_0 \leqslant \sup_{x \in \mathbb{R}^M} |\nabla g(x)|$$

Since g is 0-standard, we obtain (4.3) [under the assumption (4.4) which gives $\rho_1 \geqslant 1$] in the special case $L = \varnothing$, $k = 1$.

Step 2. Let $L = \varnothing$, $k \geqslant 2$ and assume that (4.3) and (4.4) have been established for $L = \varnothing$ and for k replaced by any $\tilde{k} < k$. Now (4.10) reads

$$\begin{aligned}
&\langle \nabla^k g, t_1 \otimes \cdots \otimes t_k \rangle \\
&\quad = (\nabla \Phi \cdot \partial_x - h\Delta)(\langle \nabla^k f, t_1 \otimes \cdots \otimes t_k) + k \langle \nabla^k f, t_1 \otimes \cdots \otimes t_k \rangle \\
&\qquad + \sum_{j=1}^{k} \langle \nabla^k f, t_1 \otimes \cdots \otimes [\text{Hess}\, \Phi(x) - I]\, t_j \otimes \cdots \otimes t_k \rangle \\
&\qquad + \sum_{\substack{\mathcal{J} \cup \mathcal{K} = \{1,\dots,k\},\, \mathcal{J} \cap \mathcal{K} = \varnothing \\ \# \mathcal{K} \leqslant k-2}} \langle t_{\mathcal{J}}(\partial_x)\, \nabla \Phi, t_{\mathcal{K}}(\partial_x)\, \nabla f \rangle \qquad (4.12)
\end{aligned}$$

Let $1=\sum(1/p_j)$, $1\leqslant\prod_{j=1}^{k}\rho_j$, $(\rho_1,\ldots,\rho_k)\in\mathscr{R}_k$, which is (4.4) in this case. In the last sum we have $\mathscr{J}\neq\varnothing$, so applying the induction hypothesis and (4.6), we obtain

$$|t_{\mathscr{J}}(\partial_x)\nabla\Phi|_{p,\rho}=\mathcal{O}(1)\prod_{j\in\mathscr{J}}|t_j|_{p_j,\rho_j}$$

$$\text{if}\quad \frac{1}{p}=\sum_{j\in\mathscr{J}}\frac{1}{p_j},\quad \rho\leqslant\prod_{j\in\mathscr{J}}\rho_j,\quad \left(\frac{1}{\rho},(\rho_j)_{\mathscr{J}}\right)\in\mathscr{R}_{1+\#\mathscr{J}}$$

$$|t_{\mathscr{K}}(\partial_x)\nabla f|_{q,1/\rho}=\mathcal{O}(1)\prod_{j\in\mathscr{K}}|t_j|_{p_j,\rho_j}$$

$$\text{if}\quad \frac{1}{q}=\sum_{j\in\mathscr{K}}\frac{1}{p_j},\quad 1\leqslant\rho\cdot\prod_{j\in\mathscr{K}}\rho_j,\quad (\rho,(\rho_j)_{\mathscr{K}})\in\mathscr{R}_{1+\#\mathscr{K}}$$

A possible choice of ρ is $\rho=\prod_{j\in\mathscr{J}}\rho_j$, and we see that

$$\langle t_{\mathscr{J}}(\partial_x)\nabla\Phi,\, t_{\mathscr{K}}(\partial_x)\nabla f\rangle=\mathcal{O}(1)\prod_{j=1}^{k}|t|_{p_j,\rho_j}$$

Using the maximum principle, we can then conclude as before, introducing

$$m_0=\sup_{(x\in\mathbb{R}^m,\ |t_j|_{p_j,\rho_j}=1)}\langle\nabla^k f(x),\, t_1\otimes\cdots\otimes t_k\rangle$$

Step 3. Let $L\neq\varnothing$ and assume that (4.3) [under the assumption (4.4)] has been established with k, L replaced by $\tilde{k}$, $\tilde{L}$, with $\tilde{L}\subset L$, $\tilde{L}\neq L$, and no restriction on $\tilde{k}$. We shall then establish (4.3) for $k=1$. (4.10) becomes

$$\begin{aligned}t_1(\partial_x)\,\partial_s^L g=&(\nabla\Phi\cdot\partial_x-h\Delta)(\langle\nabla\partial_s^L f,t_1\rangle)\\&+\langle\nabla\partial_s^L f,t_1\rangle+\langle\nabla\partial_s^L f,(\operatorname{Hess}\Phi-I)t_1\rangle\\&+\sum_{\substack{\tilde{\mathscr{J}}\cup\tilde{\mathscr{K}}=L,\ \tilde{\mathscr{J}}\cap\tilde{\mathscr{K}}=\varnothing\\ \tilde{\mathscr{K}}\neq L}}\langle t_1(\partial_x)\,\partial_s^{\tilde{\mathscr{J}}}\nabla\Phi,\,\partial_s^{\tilde{\mathscr{K}}}\nabla f\rangle\\&+\sum_{\substack{\tilde{\mathscr{J}}\cup\tilde{\mathscr{K}}=L,\ \tilde{\mathscr{J}}\cap\tilde{\mathscr{K}}=\varnothing\\ \tilde{\mathscr{K}}\neq L}}\langle\partial_s^{\tilde{\mathscr{J}}}\nabla\Phi,\,t_1(\partial_x)\,\partial_s^{\tilde{\mathscr{K}}}\nabla f\rangle\end{aligned}\tag{4.13}$$

According to (4.4), we shall take

$$p_1=1,\qquad 1\leqslant\left(\prod_{l\in L}\tilde{\rho}_l\right)\rho_1,\qquad \tilde{\rho}_l\leqslant\hat{\rho}_l,\qquad ((\tilde{\rho}_l)_{l\in L},\rho_1)\in\mathscr{R}_{(\#L+1)}$$

The general term of the second to last sum in the r.h.s. of (4.13) can be rewritten as

$$\begin{aligned}&\langle t_1(\partial_x)\,\partial_s^{\tilde{\mathscr{J}}}\,\nabla\Phi,\,\partial_s^{\tilde{\mathscr{K}}}\,\nabla f\rangle\\&\quad=\langle\nabla^2\partial_s^{\tilde{\mathscr{J}}}\,\Phi,\,t_1\otimes\partial_s^{\tilde{\mathscr{K}}}\,\nabla f\rangle=\mathcal{O}(1)\,|t_1|_{1,\rho_1}\,|\partial_s^{\tilde{\mathscr{K}}}\,\nabla f|_{\infty,\rho_1'}\\&\qquad\text{if}\quad 1\leqslant\Big(\prod_{j\in\tilde{\mathscr{J}}}\tilde{\rho}_j\Big)\,\rho_1\rho_1',\quad((\tilde{\rho}_j)_{j\in\tilde{\mathscr{J}}},\rho_1,\rho_1')\in\mathscr{R}_{(\#\tilde{\mathscr{J}}+2)}\end{aligned}$$

On the other hand, by the induction hypothesis

$$|\partial_s^{\tilde{\mathscr{K}}}\,\nabla f|_{\infty,\rho_1'}=\mathcal{O}(1)\qquad\text{if}\quad 1\leqslant\Big(\prod_{j\in\tilde{\mathscr{K}}}\tilde{\rho}_j\Big)\frac{1}{\rho_j'},\quad\Big((\tilde{\rho}_j)_{j\in\tilde{\mathscr{K}}},\frac{1}{\rho_1'}\Big)\in\mathscr{R}_{(\#\tilde{\mathscr{K}}+1)}$$

The choice $\rho_1'=\prod_{j\in\tilde{\mathscr{K}}}\tilde{\rho}_j$ satisfies the requirement above and we conclude that the second to last sum in (4.13) is $\mathcal{O}(1)\,|t_1|_{1,\rho_1}$. In the same way, we can estimate the last sum, and using the maximum principle, we obtain

$$\langle\nabla\partial_s^L f,\,t_1\rangle=\mathcal{O}(1)\,|t_1|_{1,\rho_1}$$

Step 4. Let $L\neq\varnothing$, $k\geqslant 2$, and assume that (4.3) [under the assumption (4.4)] has been established for all $\tilde{k}$, $\tilde{L}$, with $\tilde{L}\subset L$, $\tilde{L}\neq L$, and for all $\hat{k}$, L, with $1\leqslant\hat{k}\leqslant k-1$.

Take $\tilde{\rho}_j$, ρ_j, p_j as in (4.4). We shall estimate the general term in the last sum in (4.10). For suitable (p,ρ) and with $1/p+1/p'=1$, the modulus of this term can be estimated by

$$|t_{\mathscr{J}}(\partial_x)\,\partial_s^{\tilde{\mathscr{J}}}\,\nabla\Phi|_{p,\rho}\,|t_{\mathscr{K}}(\partial_x)\,\partial_s^{\tilde{\mathscr{K}}}\,\nabla f|_{p',1/\rho}$$

Moreover, since $\mathscr{J}\neq\varnothing$ when $\tilde{\mathscr{J}}=\varnothing$,

$$\begin{aligned}&|t_{\mathscr{J}}(\partial_x)\,\partial_s^{\tilde{\mathscr{J}}}\,\nabla\Phi|_{p,\rho}\\&\qquad=\mathcal{O}(1)\prod_{j\in\mathscr{J}}|t_j|_{p_j,\rho_j}\qquad\text{if}\quad\begin{cases}\dfrac{1}{p}=\displaystyle\sum_{j\in\mathscr{J}}\frac{1}{p_j}\\ \Big((\tilde{\rho}_j)_{\tilde{\mathscr{J}}},(\rho_j)_{\mathscr{J}},\dfrac{1}{\rho}\Big)\in\mathscr{R}_{\#\tilde{\mathscr{J}}+\#\mathscr{J}+1}\\ 1\leqslant\Big(\displaystyle\prod_{j\in\tilde{\mathscr{J}}}\tilde{\rho}_j\Big)\Big(\prod_{j\in\mathscr{J}}\rho_j\Big)\frac{1}{\rho}\end{cases}\end{aligned}$$

$$\begin{aligned}&|t_{\mathscr{K}}(\partial_x)\,\partial_s^{\tilde{\mathscr{K}}}\,\nabla f|_{p',1/\rho}\\&\qquad=\mathcal{O}(1)\prod_{j\in\mathscr{K}}|t_j|_{p_j,\rho_j}\qquad\text{if}\quad\begin{cases}\dfrac{1}{p'}=\displaystyle\sum_{j\in\mathscr{K}}\frac{1}{p_j}\\ ((\tilde{\rho}_j)_{\tilde{\mathscr{K}}},(\rho_j)_{\mathscr{K}},\rho)\in\mathscr{R}_{\#\tilde{\mathscr{K}}+\#\mathscr{K}+1}\\ 1\leqslant\Big(\displaystyle\prod_{j\in\tilde{\mathscr{K}}}\tilde{\rho}_j\Big)\Big(\prod_{j\in\mathscr{K}}\rho_j\Big)\rho\end{cases}\end{aligned}$$

The choice of p, p' is then clear, and with $\rho = (\prod_{j \in \tilde{\mathscr{J}}} \tilde{\rho}_j)(\prod_{j \in \mathscr{J}} \rho_j)$, the conditions involving $\tilde{\rho}_j$, ρ_j, ρ are also fulfilled. The last term in (4.10) is therefore $\mathcal{O}(1) \prod_{j=1}^{k} |t_j|_{p_j, \rho_j}$, and after that verification we can apply the maximum principle as before. ▮

Remark 4.2. If we want, for example, to have (4.3) for f corresponding to $\# L \leqslant 1$, $k \leqslant 2$, then we need only to have the same estimates on g for $\# L \leqslant 1$, $k \leqslant 2$ and the corresponding information about Φ for $\# L \leqslant 1$ and $2 \leqslant \# L + k \leqslant 4$.

5. UNIFORM ESTIMATES (THE GENERAL CASE)

We recall that under the weak assumptions (2.2), (2.3)

$$|\partial_x^\alpha \nabla \Psi| \leqslant C_\alpha \qquad \forall \alpha \in \mathbb{N}^m \tag{5.1}$$

$$|\partial_x^\alpha w| \leqslant C_\alpha \qquad \forall \alpha \in \mathbb{N}^m \tag{5.2}$$

and

$$\text{Hess } \Phi \geqslant \rho > 0 \tag{5.3}$$

(for some $0 < \rho < 1$), we have proved in Section 2.1 the existence of a C^∞ vector field v for a given w. Moreover, under the additional assumption that w and Ψ are with compact support, we have proved (2.4), that v and all its derivatives tend to zero when $|x| \to \infty$, and in Section 3.2 that v is a gradient if w is a gradient and a precise control of norms on v under suitable assumptions on w and Ψ (see Section 4). In this section, we want to relax the assumptions of compact support used in the preceding section. For this we introduce as in ref. 27 a family of cutoff functions $\chi = \chi_\varepsilon$ ($\varepsilon \in [0, 1]$) in $C_0^\infty(\mathbb{R})$ with values in $[0, 1]$ such that

$$\begin{aligned} \chi &= 1 && \text{for} \quad |t| \leqslant \varepsilon^{-1} \\ |\chi^{(k)}(t)| &\leqslant C_k \varepsilon / |t|^{(k)} && \text{for} \quad k \in \mathbb{N} \end{aligned} \tag{5.4}$$

We can take, for example, $\chi_\varepsilon(t) = f(\varepsilon \ln(|t|))$ for a suitable f. We then introduce

$$\Psi_\varepsilon(x) = \chi_\varepsilon(|x|) \Psi \tag{5.5}$$

and

$$w_\varepsilon(x) = \nabla(\chi_\varepsilon(|x|) g) \tag{5.6}$$

We first verify that the assumptions in Section 2 are uniformly satisfied for all the family of Ψ_ε and w_ε.

The Hessian of Φ_ε is uniformly minorized. We have indeed

$$\text{Hess } \Psi \geqslant \rho - 1$$

$$\text{Hess } \Psi_\varepsilon(x) \geqslant (\rho - 1)\, \chi_\varepsilon(|x|) - C\varepsilon$$

for all ε and some constant C. It follows immediately that

$$\text{Hess } \Phi_\varepsilon \geqslant (\rho - C\varepsilon)$$

If we choose $\rho' < \rho$, then we observe that the condition (5.3) is satisfied with ρ replaced by ρ' for ε small enough. Here we emphasize that our "ε small enough" is possibly m dependent. It is also easy to get the existence of $C'_\alpha > C_\alpha$, $\alpha \in \mathbb{N}^m$, such that, for all ε,

$$|(\partial_x^\alpha \nabla \Psi_\varepsilon)| \leqslant C'_\alpha$$

for all α and the same property is true for the family w_ε. Finally, we observe, under the assumptions (5.1)–(5.3), that the family of vector fields

$$q_\varepsilon := \exp -\frac{\Phi_\varepsilon}{2} \cdot w_\varepsilon$$

is bounded (by a possibly m-dependent constant) in $B^k(\mathbb{R}^m; \mathbb{R}^m)$ for any k.

Boundedness of u_ε. The proof given in Section 2 gives first a uniform control of the solution u_ε in B^0 (take the scalar product with u_ε and use the uniform strict convexity). We then deduce that

$$\left(-\Delta + \frac{|\nabla \Phi_\varepsilon|^2}{4}\right) u_\varepsilon = \tilde{q}_\varepsilon$$

where

$$\tilde{q}_\varepsilon = q_\varepsilon + \frac{\Delta \Phi_\varepsilon}{2} u_\varepsilon - \text{Hess } \Phi_\varepsilon\, u_\varepsilon$$

is bounded in B^0. Taking again the scalar product with u_ε, we get the uniform control of ∇u_ε and of $(\nabla \Phi_\varepsilon) u_\varepsilon$ in B^0. We then obtain easily a uniform control of u_ε in B^1. Observing now that

$$\nabla \Phi_\varepsilon - \nabla \Phi = \nabla \chi_\varepsilon \cdot \Psi + (\chi_\varepsilon - 1)\, \nabla \Psi$$

is uniformly bounded with all its derivatives, this permits us now to see that

$$\left(-\Delta + \frac{|\nabla \Phi|^2}{4}\right) u_\varepsilon = \hat{q}_\varepsilon$$

where

$$\hat{q}_\varepsilon = \tilde{q}_\varepsilon + \frac{|\nabla\Phi_\varepsilon - \nabla\Phi|^2}{4} u_\varepsilon - \frac{(\nabla\Phi_\varepsilon - \nabla\Phi)\cdot(\nabla\Phi_\varepsilon)}{2} u_\varepsilon$$

is bounded in B^0. By classical results on the harmonic oscillators we get immediately that u_ε is uniformly bounded in B^2. This property of u_ε permits us to get now that $\hat{q}_\varepsilon$ is bounded in B^1 and we can continue by a bootstrap argument. We have consequently obtained that

$$u_\varepsilon \text{ is uniformly bounded in } B^k \text{ for any } k \tag{5.7}$$

Convergence of u_ε. We shall now analyze the convergence of u_ε to u.

The first point is here to remark that

$$\|q_\varepsilon - q_0\|_{B^0} \leqslant C\varepsilon \tag{5.8}$$

We have indeed

$$q_\varepsilon - q_0 = [\exp(-\Phi_\varepsilon)(w_\varepsilon - w)] + \{[\exp(-\Phi_\varepsilon) - \exp(-\Phi)]\, w\}$$

If we observe that for a suitable constant C, we have for all ε

$$\phi_\varepsilon(x) \geqslant \frac{x^2}{4} - C$$

the two terms are easily controlled. For the first term we have the bound

$$|x|^k\, |[\exp(-\Phi_\varepsilon)(w_\varepsilon - w)](x)| \leqslant D\, |x|^k\, |w|\, (1-\chi_\varepsilon)(x) \exp\left(-\frac{x^2}{4}\right)$$

$$\leqslant D\varepsilon\, |x|^{k+1}\, |w| \exp\left(-\frac{x^2}{4}\right) \leqslant D'\varepsilon$$

For the second term, we write

$$|x|^k\, [\exp(-\Phi_\varepsilon) - \exp(-\Phi) w](x) \leqslant C\, |x|^k \exp\left(-\frac{x^2}{4}\right) |(1-\chi_\varepsilon)(x)| \cdot |\Psi(x)|$$

$$\leqslant D\varepsilon\, |x|^{k+2}\, |w| \exp\left(-\frac{x^2}{4}\right) \leqslant D'\varepsilon$$

The second point is to observe that

$$\left(-\Delta + \frac{|\nabla\Phi|^2}{4} - \Delta\Phi + \text{Hess}\,\Phi\right)(u_\varepsilon - u)$$

$$= (q_\varepsilon - q) + \left(\frac{|\nabla\Phi|^2}{4} - \frac{|\nabla\Phi_\varepsilon|^2}{4}\right) u_\varepsilon + (\Delta\Phi_\varepsilon - \Delta\Phi) u_\varepsilon$$

$$+ (\text{Hess}\,\Phi - \text{Hess}\,\Phi_\varepsilon) u_\varepsilon \tag{5.9}$$

Using that $|x|^k \cdot u_\varepsilon$ is bounded in B^0 for any k, we get easily that the r.h.s. of this last equation is $\mathcal{O}(\varepsilon)$ in B^0 and consequently that $(u_\varepsilon - u)$ is $\mathcal{O}(\varepsilon)$ in B^2. Then it is not more difficult to prove that the r.h.s. is $\mathcal{O}(\varepsilon)$ in B^k for all k, and we get the property that $(u_\varepsilon - u)$ is $\mathcal{O}(\varepsilon)$ in B^k for all k. We can now obtain:

Proposition 5.1. Under the assumptions (5.1)–(5.3), the family v_ε is convergent as $\varepsilon \to 0$ to the solution v in C^∞.

End of the Proof of the Proposition. We just observe that $v_\varepsilon = (\exp \Phi_\varepsilon/2) u_\varepsilon$ and use the convergence of u_ε to u in C^∞. We recall also that, for a given x, $\Phi_\varepsilon(x) = \Phi(x)$ for ε small enough. ∎

Once we have this weak convergence result (with no control at ∞), we shall apply the estimates of the Section 4 to v_ε.

We now assume that g is 0-standard ($w = \nabla g$) and that Φ satisfies (4.6)–(4.7). In particular, the conditions (5.1)–(5.3) are satisfied. We explain for the moment the case without parameter. Let us consider as before the approximate family $(g_\varepsilon, \Phi_\varepsilon)$. As in ref. 27, we observe now that the approximate family satisfies the same estimates modulo $\mathcal{O}(\varepsilon)$. This $\mathcal{O}(\varepsilon)$ is uniform with respect to x, but may depend on the dimension. Let us just give an example of the technique:

$$w_\varepsilon = \chi_\varepsilon(|x|)\, w + \chi'_\varepsilon(|x|)\, g(x)\, x/|x|$$

We observe now that $g(x) = x \cdot \int_0^1 w(tx)\, dt$ with the assumption $g(0) = 0$. If B is a normed space with underlying space $\mathbb{R}^m$, we get

$$\sup_x |w_\varepsilon(x)|_B \leqslant (\sup_x |w(x)|_B) + C(\varepsilon)(\sup_x |w(x)|_B)$$

More generally, if g is 0-standard, we get

$$\sup_x \langle \nabla^k u, t_1 \otimes \cdots \otimes t_k \rangle = [\mathcal{O}_{k, \# L}(1) + \varepsilon \mathcal{O}_{(L, \rho, \rho')}(1)] \prod_{j=1}^{k} |t_j|_{p_j, \rho_j} \qquad (5.10)$$

Observing that we have similar properties for the family Φ_ε and that $f_\varepsilon(x)$ tends to 0 at ∞ as proved in Section 2, we can follow the proof of Proposition 4.1, and we get uniform estimates for f_ε modulo $\mathcal{O}(\varepsilon)$. The weak convergence result of f_ε to f in C^∞ permits us to conclude that f is 0-standard. Finally, we have proved:

Proposition 5.2. Assume that g is 0-standard, and that Φ satisfies (4.6), (4.7) and has the form $x^2/2 + \Psi(x)$ with $\nabla \Psi(x)$ bounded; then the solution f of (4.5) is 0-standard.

Remark 5.3. If $\Phi(x, s) = x^2/2 + \Psi(x, s)$ and $\nabla\Psi(x, s)$ is bounded in (x, s), then Proposition 5.2 is true in the parameter sense. The weak dependence with respect to the parameter permits us indeed to prove an analog of Proposition 5.1 with parameters. The proof is left to the reader.

Remark 5.4. The notion of 0-standard is very convenient because it permits us to control all the derivatives, but by looking at the proof we can also give minimal assumptions in order to control some specified derivative of f. We shall sometimes use this remark in order to give weaker assumptions for our statements. For example, the proof of Theorem 1.6 with minimal assumptions will be a consequence of Remark 4.2.

6. THE THERMODYNAMIC LIMIT

In this section we shall analyze more precisely the existence of the thermodynamic limit and control the speed of convergence of the limit in the spirit of ref. 27.

6.1. Proof of Proposition 1.1

We recall that we are considering a family

$$\Phi^{(m)}(x) = \frac{x^2}{2} + \Psi^{(m)}(x)$$

satisfying the following properties:

$$\Psi^{(m)}(0) = 0 \tag{6.1}$$

$$|\nabla_x \Psi^{(m)}|_{l^\infty} \leqslant C \tag{6.2}$$

$$|\nabla_x(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))|_{l^1} \leqslant C \tag{6.3}$$

$$|\Delta(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))| \leqslant C \tag{6.4}$$

In order to get the existence of the thermodynamic limit, it is sufficient to prove a result of "approximative additivity." It is sufficient to prove that

$$|a(m+n) - a(m) - a(n)| < D$$

for some D, where we recall that

$$a(m) = -\ln\left\{\int \exp[-\Phi^{(m)}(x)/h]\, dx\right\}$$

The assumptions given here seem the optimal if we follow the proof proposed in ref. 12. We introduce the family

$$\Phi^{(m,n)}(x, s) = (1-s)\,\Phi^{(m+n)}(x) + s[\Phi^{(m)}(x') + \Phi^{(n)}(x'')] \tag{6.5}$$

with $x = (x', x'') \in \mathbb{R}^{n+m}$; $s \in [0, 1]$.

We shall study the logarithmic derivative of

$$\int_{\mathbb{R}^{m+n}} \exp -[\Phi^{(m,n)}(x, s)/h]\, dx$$

with respect to s, which is given by

$$-\frac{1}{h}\int_{\mathbb{R}^{m+n}} [\partial_s(\Phi^{(m,n)}(x, s)] \exp -[\Phi^{(m,n)}(x, s)/h]\, dx^{(m+n)} \tag{6.6}$$

We have just to prove a uniform bound on this logarithmic derivative with respect to $s \in [0, 1]$, m, n. We have

$$\partial_s \Phi^{(m,n)}(x, s) = (\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))$$

and

$$\begin{aligned}
&(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(x)\\
&\quad = \sum_{l=1}^{m+n} \left\{ x_l \int_0^1 [\partial_{x_l}(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(tx)]\, dt \right\}
\end{aligned} \tag{6.7}$$

An integration by part gives

$$\begin{aligned}
&-\frac{1}{h}\left\{\int_{\mathbb{R}^{m+n}} (\partial_s \Phi^{(m,n)}) \cdot \exp -[\Phi^{(m,n)}(x, s)/h]\, dx\right\}\\
&\quad = -\left\{\int_{[0,1]} \int_{\mathbb{R}^{m+n}} \Delta(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(tx)\right.\\
&\qquad \left. \times \exp -[\Phi^{(m,n)}(x, s)]\, t\, dx\, dt\right\}\\
&\qquad + \frac{1}{h}\left(\int_{[0,1]} \int_{\mathbb{R}^{m+n}} [\nabla(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(tx)]\right.\\
&\qquad \times \{[(1-s)\,\nabla\Psi^{(m+n)} + s\nabla(\Psi^{(m)} \oplus \Psi^{(n)})]\}(x)\\
&\qquad \left. \times \exp -[\Phi^{(m,n)}(x, s)/h]\, dx\, dt)\right)
\end{aligned}$$

We can then use the uniform estimates given by the assumptions and we get the Proposition. We observe that the constant D with respect to h is $\mathcal{O}(1/h)$. ▮

Remark 6.1. "Boundary Effect." This very explicit proof permits us to see that we can perturb the function $\Psi^{(m)}(x)$ by adding a term of the form: $\psi^{(m)}$ satisfying the conditions

$$\begin{aligned}
&\text{(a)} \quad \psi^{(m)}(0)=0\\
&\text{(b)} \quad \sup_x |\nabla\psi^{(m)}(x)|_{l^1}=o(1)m\\
&\text{(c)} \quad \sup_x |\Delta\psi^{(m)}|=o(1)m\\
&\text{(d)} \quad \sup_x |\nabla\psi^{(m)}|_{l^2}=o(1)m^{1/2}
\end{aligned} \tag{6.8}$$

Using Hölder, (d) is, for example, a consequence of (b) and

$$\text{(e)} \quad \sup_x |\nabla\psi^{(m)}|_{l^\infty}=\mathcal{O}(1)$$

Under these conditions the thermodynamic limit is unchanged by the addition of $\psi^{(m)}$.

6.2. The General Strategy

We introduce first a new parameter t and consider the family of potentials

$$\Phi^{(m)}(x,t)=\frac{x^2}{2}+t\Psi^{m}(x); \qquad t\in[0,1]$$

and we are interested in the behavior with respect to m of the quantity

$$a(m,t)=-\ln\left\{\int \exp[-\Phi^{(m)}(x,t)/h]\,dx\right\}$$

for $t=1$.

We shall see that the family $\Phi^{(m)}(x,t)$ satisfies all the assumptions of Section 5 in the t-dependent sense. Because $a(m,0)$ is well known and we are interested in $a(m,1)$, it is natural to study the derivative with respect to t, $a'(m,t)$:

$$a'(m,t)=\frac{1}{h}\langle(\partial_t\Phi^{(m)}(\cdot,t))\rangle_{\Phi^{(m)}} \tag{6.9}$$

and we get the information on $a(m, 1)$ by integration over $[0, 1]$ with respect to t. We are then reduced to the proof that $a'(m, t)/m$ converges exponentially rapidly with respect to m to a "thermodynamic limit" and uniformly with respect to t. We forget now an explicit reference to m and introduce

$$b(t) = a'(m, t)$$

Following an idea of ref. 29, we are looking for a vector field $v(x, t)$ with temperate coefficients such that

$$(\partial_t \Phi)(x, t) = b(t) + v(x, t) \cdot \nabla_x \Phi(x, t) - \operatorname{div}_x v(x, t) \tag{6.10}$$

and observe that if Φ and v satisfy this equation, then

$$b = -a'(m, t)/h$$

If we find such a vector field $v(x, t)$, then we get an expression for $b(t)$ by observing that at the (unique) critical point $x(t)$ of Φ we have

$$b(t) = (\partial_t \Phi)(x(t), t) - (\operatorname{div}_x v)(x(t), t) \tag{6.11}$$

Good estimates on $\operatorname{div}_x v$ and on $(\partial_t \Phi)(x(t), t)$ will give good estimates on $b(t)$. If we define

$$g(x, t) = (\partial_t \Phi)(x, t)$$

and differentiate Eq. (6.10) with respect to x, we get, as observed in Section 2, the "basic" equation

$$\nabla_x g(x, t) = [-\Delta + (\nabla_x \Phi)(x, t) \cdot \nabla_x]\, v(x, t) + \operatorname{Hess}_x \Phi(x, t)\, v(x, t) \tag{6.12}$$

(assuming v to be a gradient), which was studied in the preceding sections.

Let us also observe that in our example

$$(\nabla_x g)(x, t) = (\nabla_x \Psi)(x)$$

and that we have assumed in Theorem 1.6 that $\nabla \Psi^{(m)}(0) = 0$, which implies, using the convexity of Φ,

$$x(t) = 0$$

We then get in this case $b(t)$ by the formula

$$b(t) = -(\operatorname{div}_x v)(t, 0)$$

6.3. Remarks on the Critical Points

Here we want to relax the assumption that $x(t) = 0$. This appears useful in the study of the magnetization and also in the study of the Schrödinger operator.

For this we first observe that $x(0) = 0$ and it is consequently sufficient to study $x'(t)$. Recall that $x(t)$ is a solution of the equation

$$\nabla_x \Phi(x(t), t) = 0 \tag{6.13}$$

Differentiating with respect to t, we get

$$(\mathrm{Hess}_x \Phi) \cdot (dx/dt)(t) = -(\partial_t \nabla_x \Phi)(x(t), t) = -(\nabla \Psi)(x(t)) \tag{6.14}$$

In particular, due to (1.9) and (1.23),

$$(dx/dt)(t) \in l^\infty \tag{6.15}$$

We can now ask for convergence properties of this critical point as m tends to ∞. The proof can also be considered as a preparation in order to understand in a simpler context what is going on in the next subsection. Let us also mention that in a related context, the study of the critical points was made in ref. 29. We introduce the family [cf. (6.5)] $\Phi^{(m,n)}(x, t, s)$, depending on two parameters $(t, s) \in [0, 1]^2$, indexed by the two integers (m, n), and defined on $\mathbb{R}^{m+n}$ by

$$\Phi^{(m,n)}(x, t, s) = \frac{x^2}{2} + t\{(1-s)\, \Psi^{(m+n)}(x) + s[\Psi^{(m)}(x') + \Psi^{(n)}(x'')]\}$$

$$\text{for} \quad x = (x', x'') \in \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \tag{6.16}$$

More briefly, we shall write sometimes

$$\Phi^{(m,n)}(x, t, s) := \frac{x^2}{2} + \Psi^{(m,n)}(x, t, s) := \frac{x^2}{2} + t\Psi^{(m,n)}(x, s)$$

We work under the assumptions given in Theorem 1.6, but we need much less, as can be seen from the analysis of the proof we give now. Let us start again from the formula (6.13), which depends now on the new parameter s and the critical point is now denoted by $x^{(m,n)}(t, s)$. Differentiating with respect to s, we get

$$(\mathrm{Hess}_x \Phi^{(m,n)})(x(t, s), t, s)(\partial_s x(t, s)) = -t(\nabla_x(\Psi^{(m+n)} - \Psi^{(m)} \oplus \Psi^{(n)})) \tag{6.17}$$

This gives us a good control of $(\partial_s x(t, s))$ in B using the assumptions (1.30) and (1.29). This means that there exists a constant C such that for any m, n, t, s, and weight $\rho_{m,n}$ we have

$$\|\partial_s x^{(m,n)}(t, s)\|_{l^\infty_{\rho_{m,n}}} \leqslant C \tag{6.18}$$

Differentiating with respect to t gives

$$(\text{Hess}_x\, \Phi^{(m,n)}(x(t, s), t, s)(\partial_t x(t, s)) = -(\nabla_x \Psi^{(m,n)})(x(t, s), s) \tag{6.19}$$

This gives us a good control of $(\partial_t x(t, s))$ in B_1

$$\|\partial_t x^{(m,n)}(t, s)\|_{l^\infty} \leqslant C \tag{6.20}$$

For $s = 0$, we have $x^{(m,n)}(t, 0) = x^{(m+n)}(t)$, and for $s = 1$, we get

$$x^{(m,n)}(t, 1) = (x^{(m)}(t), x^{(n)}(t)) \in \mathbb{R}^m \times \mathbb{R}^n$$

We now differentiate (6.19) with respect to s and we get for any vector v in $\mathbb{R}^{(m+n)}$

$$\begin{aligned} &\langle(\text{Hess}_x\, \Phi^{(m,n)})(x(t, s), t, s)(\partial^2_{t,s} x(t, s)), v\rangle \\ &\quad = -\langle(\text{Hess}_x\, \partial_s \Phi^{(m,n)})(x(t, s), t, s)(\partial_t x(t, s)), v\rangle \\ &\qquad -\langle(\nabla^3_x \Phi^{(m,n)})(x(t, s), t, s), (\partial_t x(t, s)) \otimes (\partial_s x(t, s)) \otimes v\rangle \\ &\qquad -\langle(\nabla_x \partial_s \Psi^{(m,n)})(x(t, s), t, s), v\rangle \\ &\qquad -\langle(\nabla^2_x \Psi^{(m,n)})(x(t, s), t, s), (\partial_s x)(t, s) \otimes v\rangle \end{aligned} \tag{6.21}$$

with $x(t, s) = x^{(m,n)}(t, s)$.

Using these computations and (1.29), (1.28), (6.25), (1.30), and (1.31), we get that

$$|(\partial^2_{ts} x^{(m,n)})(t, s)|_{l^\infty_{\rho_{m,n}}} \leqslant C \tag{6.22}$$

In particular, we obtain for any $i \in \{1,..., m\}$ and any $n \geqslant 0$

$$\begin{aligned} |x_i^{(m+n)}(1) - x_i^{(m)}(1)| &= \left| \int_0^1 \int_0^1 (\partial^2_{ts} x_i^{(m,n)})(t, s)\, dt\, ds \right| \\ &\leqslant C \exp -\kappa \inf(i, m-i) \end{aligned} \tag{6.23}$$

But we have seen that $x_i^{(m+n)}(1)$ is independent of i and we get immediately for any $i \in \{1,..., m\}$ and any $n \geqslant 0$

$$|x_i^{(m+n)}(1) - x_i^{(m)}(1)| \leqslant C \exp -\kappa m/2$$

This proves that $m \to x_i^{(m)}$ is a convergent sequence and that the convergence is exponentially rapid. We have finally proved the following:

Proposition 6.2. Let $\Phi^{(m)}$ be a family of potentials satisfying the assumptions of Proposition 1.1, (1.22), (1.24), (1.29), (1.28), (6.25), (1.30), and (1.31). Then there exists $y \in \mathbb{R}$ s.t. the critical point $x^{(m)}$ of $\Phi^{(m)}$ satisfies for all i the property

$$|x_i^{(m)} - y| \leqslant C \exp -\kappa m/2$$

This proposition will be sufficient for most of the problems we shall meet, but if we return to the problem of the study of the thermodynamic limit we will have to compute the quantity $\Psi^{(m)}(x^{(m)}(1))$ and need to prove that $\Psi^{(m)}(x^{(m)}(1))/m$ is convergent. In order to prove convergence, we consider again the family $\psi^m(t) = \Psi^{(m)}(x^{(m)}(t))$ and we observe that

$$\psi^m(0) = 0$$

and that

$$d\psi^m(t)/dt = \nabla_x \Psi^{(m)}(x^{(m)}(t)) \cdot (dx^{(m)}(t)/dt)$$

and using the fact that $(x^{(m)}(t))$ is the critical value, we get

$$\begin{aligned} d\psi^m(t)/dt &= -\{x^{(m)}(t) \cdot [dx^{(m)}(t)/dt]\}/t \\ &= -(m/2t)\, x_i^{(m)} \cdot \{d[x_i^{(m)}(t)]/dt\} \end{aligned}$$

for any i (we have used here the invariance by circular permutation).

We know already that $x_i^{(m)}(t)$ converges exponentially rapidly (and uniformly with respect to $t \in [0, 1]$) to a limit $y(t)$. We have just to prove the convergence of $d(x_i^{(m)}(t))/dt$ for fixed i as m tends to ∞. But the proof is similar to what we do in order to obtain (6.23) and we get

$$\begin{aligned} &|(dx_i^{(m+n)}/dt)(t) - (dx_i^{(m)}/dt)(t)| \\ &\quad = \left| \int_0^1 (\partial_{ts}^2 x_i^{(m,n)})(t, s)\, ds \right| \leqslant C \exp -\kappa \inf(i, m-i) \end{aligned} \tag{6.24}$$

We get, for a convenient choice of i ($i = [m/2]$), the uniform exponential convergence with respect to t of $m \to (dx_i^{(m)}/dt)(t)$, which implies also the uniform exponential convergence with respect to t of $m \to (x_i^{(m)})(t)/t$. Finally we have proved:

Proposition 6.3. Under the assumptions of Proposition 6.2, the function $\Phi^{(m)}(x^{(m)})/m$ converges exponentially rapidly to some $\phi_\infty \in \mathbb{R}$.

Remark 6.4. It is clear that we have the same property for $\Psi^{(m)}(x^{(m)})/m$. Let us only recall that

$$(x^{(m)})^2/m = (x_i^{(m)}), \qquad \forall i = 1,..., m$$

6.4. Proof of the Existence of the Thermodynamic Limit

We first give the complete assumptions of Theorem 1.6. With the same notations we add

$$\|\nabla^3(\Phi^{(m,n)})(x,s)\|_{\mathscr{L}(B_1\times B, B)} \leqslant C \quad \text{for any} \quad B \in \mathscr{B}_2 \quad (6.25)$$

$$\|\nabla^4(\Phi^{(m,n)})(x,s)\|_{\mathscr{L}(B_1\times B_1\times B, B)} \leqslant C \quad \text{for any} \quad B \in \mathscr{B}_2 \quad (6.26)$$

$$\|\nabla^3(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))(x)\|_{\mathscr{L}(B_1\times B_1, B)} \leqslant C \quad \text{for any} \quad B \in \mathscr{B}_2 \quad (6.27)$$

We observe first that the assumptions given in Theorem 1.6 permit us to verify the assumptions of Proposition 5.2, or, more precisely, the modified version of this proposition suggested in Remarks 4.2 and 5.4. Let us consider again the family (cf. Section 6.2) with two parameters $(s_1, s_2) \in [0, 1]^2$ (but we have changed the notations in order to use the results of Section 4):

$$\Phi^{(m,n)}(x, s_1, s_2) = \frac{x^2}{2} + s_1\{(1-s_2)\, \Psi^{(m+n)}(x) + s_2[\Psi^{(m)}(x') + \Psi^{(n)}(x'')]\} \tag{6.28}$$

The corresponding family $g^{(m,n)}$ is defined by

$$\begin{aligned} g^{(m,n)}(x, s_1, s_2) &= \Psi^{(m,n)}(x, s_2) \\ &= \{(1-s_2)\, \Psi^{(m+n)}(x) + s_2[\Psi^{(m)}(x') + \Psi^{(n)}(x'')]\} \end{aligned} \tag{6.29}$$

and we are looking at the equation [see (6.12)]

$$\begin{aligned} \nabla_x \Psi^{(m,n)}(x, s_2) &= \nabla_x g^{(m,n)}(x, s_2) \\ &= [-\Delta + (\nabla_x \Phi^{(m,n)}) \cdot \nabla_x]\, v^{(m,n)}(x, s_1, s_2) \\ &\quad + \mathrm{Hess}_x\, \Phi^{(m,n)} v^{(m,n)}(x, s_1, s_2) \end{aligned} \tag{6.30}$$

with

$$v^{(m,n)}(x, s_1, s_2) = \nabla_x f^{(m,n)}(x, s_1, s_2)$$

Let us define now the corresponding weights in the sense of Section 4. Our family of weights is the following. The class $\mathscr{R}$ is the class of constant multiples of weights satisfying (1.27) and (1.18), but in this application we shall only use $\rho_{m,n} = 1$. But we shall associate to the two extra parameters (s_1, s_2) the following family of weights:

$$\hat{\rho}_1 = 1 \tag{6.31}$$

and

$$\begin{aligned} \hat{\rho}_2(j) &= \hat{\rho}_{m,n}(j) = 1 && \text{for} \quad m \leqslant j \leqslant m+n \\ \hat{\rho}_2(j) &= \hat{\rho}_{m,n}(j) = \exp \kappa \inf(j-1, m-j) && \text{for} \quad 1 \leqslant j \leqslant m \end{aligned} \tag{6.32}$$

We observe now that according to the assumption (1.22) and due to the uniqueness of the solution v, we get (with $t = s_1$)

$$\operatorname{div} v^{(m)}(0, t)/m = (\partial_i v_i^{(m)})(0, t); \qquad \forall i = 1,..., m \tag{6.33}$$

This permits us, in particular, in order to estimate

$$b(m, t) = (\partial_i v_i^{(m)})(0, t)$$

to choose i depending on m. In order to get a control of the convergence, we shall bound $(b(m+n, t) - b(m, t))$, observing that

$$b(m+n, s_1) - b(m, s_1) = \int_0^1 (\partial^2_{i,s_2} v_i^{(m,n)})(0, s_1, s_2)\, ds_2$$

for any $i \in \mathbb{N}$ s.t. $1 \leqslant i \leqslant m$.

We observe now the inequality

$$\begin{aligned} &|(\partial^2_{i,s_2} v_i^{(m,n)})(0, s_1, s_2)| \\ &\qquad \leqslant [\exp -\kappa \inf(i, m-i)]\big(\sup_{(x,s_1,s_2)} |(\partial^2_{i,s_2} v_i^{(m,n)})(x, s_1, s_2)|_{\mathscr{L}(l^\infty, l^\infty_{\hat{\rho}_2})}\big) \end{aligned}$$

The choice of $i = [m/2]$ and the proof of Proposition 5.2 gives finally

$$|(b(m+n, t)/(m+n)) - (b(m, t)/m)| \leqslant D \exp -(\kappa m/2) \tag{6.34}$$

This proves that $(b(m, t)/m)$ is a Cauchy sequence (uniformly with respect to t) and taking the limit $n \to \infty$ in (6.34), we get finally

$$|(\lim_{p \to \infty} b(p, t)/p) - (b(m, t)/m)| \leqslant D \exp -(\kappa m/2). \quad \blacksquare \tag{6.35}$$

Remark 6.5. According to what is proved in Section 6.3, Theorem 1.6 is valid without assumption (1.21). As mentioned between (6.11) and (6.12), we have just to control the terms $\partial_t \Phi^{(m)}(x^{(m)}(t), t)/m =$

$\Psi^{(m)}(x^{(m)}(t))/m$, which was the object of Proposition 6.3, and the term $(\mathrm{div}_x\, v^{(m)})(x^{(m)}(t), t)/m$, which was already proved [after a translation by $x^{(m)}(t)$].

7. PROPERTIES OF THE LIMIT MEASURES

In this section we shall study more precisely the family of measures $\mu^{(m)}$ introduced in Section 1. In particular we shall analyze the behavior of these measures when restricted to functions depending only on a fixed (independent of m) number of variables.

7.1. Existence of the Limit Mean Value

We shall prove in this subsection Proposition 1.7. We recall that we consider a C^∞ function c defined on $\mathbb{R}^k$ s.t. $\nabla c \in C_b^\infty(\mathbb{R}^k)$ that we identify with a function on $\mathbb{R}^m$

$$c^{(m)}(x) = c(x_1, x_2, \ldots, x_k) \tag{7.1}$$

and we are interested in the behavior of the mean value $\langle c^{(m)} \rangle_m$ of c with respect to the measure $\mu^{(m)}$ as m tends to ∞. We shall only analyze the convergence of $\langle c^{(m)} \rangle_m$ and the rate of convergence. As in the case of the study of the thermodynamic limit in Section 6, we arrive naturally at the study of the equation

$$c^{(m)}(x) = b + v(x) \cdot \nabla_x \Phi(x) - \mathrm{div}_x\, v(x) \tag{7.2}$$

and we observe that

$$b = \langle c^{(m)} \rangle_m \tag{7.3}$$

and also that

$$c^{(m)}(0) = c(0) = b - \mathrm{div}_x\, v(0) \tag{7.4}$$

Here we use the assumption that $\Psi^{(m)}(0) = \nabla \Psi^{(m)}(0) = 0$. As before, we have to estimate $\mathrm{div}_x\, v(0)$, but under other conditions. In particular, we have partially lost the invariance by circular permutation which was very useful in order to get the existence of the thermodynamic limit. What we need is to prove the convergence of $(\mathrm{tr\, Hess}\, f)(0)$, which usually is a consequence of estimates in $\mathscr{L}(l^\infty, l^1)$ or in $\mathscr{L}(l^\infty, l^\infty_\rho)$. A more detailed study will be given in Part II. We concentrate here on the proof of the convergence.

As for the study of the thermodynamic limit, we replace Eq. (7.2) by

$$w = (\nabla\Phi\nabla - \Delta)v + \text{Hess } \Phi v \tag{7.5}$$

where $w = \nabla c$, $c = c^{(m)}$, and $v = \nabla f$. In order to apply the results of Section 5, we first observe that $\nabla c^{(m)}$ is bounded in B_2 with $B_2 = l^\infty_{\hat\rho}$ and $\hat\rho = \rho^m_k$ given by

$$\rho^m_k(j) = \exp[\kappa d(j, \{1,..., k\})]$$

where $d(j, \{1,..., k\})$ measures the distance of j to the set $\{1,..., k\}$ in $\mathbb{Z}/m\mathbb{Z}$.

We have similar properties for the derivatives of c.

We then obtain, through the proof of Proposition 5.2, in a case without parameters, the existence of v with the following properties: $v^{(m)}$ is bounded in B_2 and $\nabla v^{(m)} = \text{Hess } f^{(m)}$ is bounded in $\mathscr{L}(B_1, B_2)$. This gives in particular that the map

$$m \to \text{tr Hess } f^{(m)}(0)$$

is bounded uniformly. This is not surprising if one thinks of the expression of the quantity as an integral. But our problem is here to prove *convergence*. We observe here that the proofs of convergence seem to be usually based in the classical literature on monotonicity arguments and correlation inequalities. We proceed here differently and we hope that this new approach will give more applications. In order to prove the convergence, we have to control

$$\text{tr Hess } f^{(m+n)}(0) - \text{tr Hess } f^{(m)}(0)$$

in order to prove that tr Hess $f^{(m)}(0)$ is a Cauchy sequence. Note that, for the moment, we have not used the invariance by circular permutation, which is in some sense broken by the choice of the identification we have made between a function of k variables c and the function $c^{(m)}$. We shall use this invariance by circular permutation in a different way, by playing with the way we realize the identification. Let us introduce, for $1 \leqslant l \leqslant (m-k)$,

$$c^{m,l}(x) = c(x_{l+1},..., x_{l+k})$$

and let us observe that

$$\langle c^{m,l} \rangle_m = \langle c^m \rangle_m$$

We choose now

$$l = [m/2]$$

and we shall compute, for $p=(m+n)$, $\langle c^p\rangle_p$ by the formula $\langle c^{p,l}\rangle_p$. By solving Eq. (7.5), with $w=\nabla c^{p,l}$, we obtain a solution $v^{(p,l)}=\nabla f^{(p,l)}$ and we are interested in the computation of

$$p\to\delta(p)\equiv \Delta f^{(p,l)}=\sum_{i=1}^{p}(\text{Hess } f^{(p,l)})_{ii}$$

and of

$$\delta(m+n)-\delta(m)$$

We can forget in the sum all the terms such that

$$d(i,[m/2])\geqslant m/4$$

because it is easy to see that the sum of all these terms is bounded by

$$\sum_{d(i,[m/2])\geqslant m/4} |(\text{Hess } f^{(p,l)})_{ii}|\leqslant D\exp(-\kappa m/8) \tag{7.6}$$

and we now have to study for $p=m$ and $p=m+n$ the terms in the sum which are "near" $m/2$.

Let us consider, as in Section 6, the family $\Phi^{(m,n)}(x,s)$ and the corresponding $v^{(m,n,l)}(x,s)$ obtained by solving Eq. (7.5) with $w=\nabla c^{m,l}$, $\Phi=\Phi^{(m,n)}(x,s)$. Here we observe that

$$\partial_s w=0$$

By the unicity argument, we have

$$v^{(m,n,l)}(x,0)=v^{(m+n,l)}(x) \qquad \text{and} \qquad v^{(m,n,l)}(x,1)=v^{(m,l)}(x')$$

In particular, we observe that for $1<i<m$

$$(\text{Hess } f^{(m,n,l)}(0,1))_{ii}$$

is independent of n. It is consequently natural to study the expression

$$\sum_{d(i,[m/2])<m/4}(\text{Hess}(\partial_s f^{(m,n,l)})(0,s))_{ii}$$

But we know from Proposition 5.2 (more precisely, from the proof of this proposition) that $(\text{Hess}(\partial_s f^{(m,n,l)})(0,s))$ is bounded [independently of (m,n)] in $\mathscr{L}(B_1,B_2)$ with

$$B_1=l^\infty, \qquad B_2=l^\infty_{\hat\rho}$$

and $\hat\rho=\rho_{m,n}$ as introduced in (6.32).

Integrating with respect to s between 0 and 1, we get

$$\left|\left(\sum_{d(i,[m/2])<m/4} (\text{Hess } f^{(m,n,l)}(0,1))_{ii}\right)-\left(\sum_{d(i,[m/2])<m/4} (\text{Hess } f^{(m,n,l)}(0,0))_{ii}\right)\right|$$
$$=\left|\left(\int_0^1 \sum_{d(i,[m/2])<m/4} (\partial_s \text{Hess } f^{(m,n,l)}(0,s))_{ii}\right)\right| \leqslant D\exp(-\kappa m/8) \qquad (7.7)$$

using the estimates coming from the proof of Proposition 5.2.

Using this with the inequality (7.6), we obtain finally the result. ∎

Remark 7.1. The result is still true [at least in the case (1.13)] if c is a function of k variables with polynomial growth. We write indeed c in the form

$$c(x) = \tilde{c}(x)\, P(x)$$

where $P(x)$ is a polynomial and where $\tilde{c}$ belongs to C_b^∞. By integration by parts we can decrease the degree of the polynomials and arrive at the bounded case.

Remark 7.2. Proposition 1.7 is still true without the assumption (1.21). According to the preceding proof, we have just to analyze the convergence of $c(x^{(m)}_{(1+l)},\ldots, x^{(m)}_{(k+l)})$, and recalling that all the components of $x^{(m)}$ are equal, we get the exponential convergence from Proposition 6.2.

7.2. Another Look at the Basic Equation

In the preceding subsection, we have seen how Eq. (7.5) appears naturally in the study of $\langle c^{(m)}\rangle_m$. This equation was studied in Section 2 in the case where $\Phi(x) = x^2/2$ for $|x|$ large. We are again interested in the integral

$$I = \int c(x)\exp[-\Phi(x)/h]\, dx$$

where c is with polynomial growth. Let us introduce [compare with (2.8)]

$$P = \left(-h\partial_x + \frac{\nabla\Phi}{2}\right)\left(h\partial_x + \frac{\nabla\Phi}{2}\right) = -h^2\Delta + \frac{|\nabla\Phi|^2}{4} - \frac{h\,\Delta\Phi}{2}$$

The first eigenvalue of P is zero and a corresponding (nonnormalized) eigenvector is $\exp(-\Phi/2h)$. If we now decompose L^2 in $\mathbb{R}\exp(-\Phi/2h) \oplus [\exp(-\Phi/2h)]^\perp$, it is clear that one can solve

$$c(x)\exp(-\Phi/2h) = P(u\exp(-\Phi/2h)) + b\exp(-\Phi/2h) \qquad (7.8)$$

with $b \in \mathbb{R}$, which is uniquely determined by

$$b\, \|\exp(-\Phi/2h)\|_{L^2}^2 = (c \exp(-\Phi/2h) \mid \exp(-\Phi/2h))$$

that is,

$$b = \frac{\int c \exp(-\Phi/h)\, dx}{\int \exp(-\Phi/h)\, dx} = \langle c \rangle$$

Of course, because

$$\exp(\Phi/2h)\, P \exp(-\Phi/2h) = (-h\nabla + \nabla\Phi) \cdot (h\nabla) = h(-h\Delta + \nabla\Phi \cdot \nabla)$$

(7.8) is equivalent to the usual equation

$$c(x) = (-h\Delta + \nabla\Phi \cdot \nabla)(hf(x)) + b \tag{7.9}$$

The operator $(-h\Delta + \nabla\Phi \cdot \nabla)$ is self-adjoint in $L^2(\mathbb{R}^m; \exp(-\Phi(x)/h)\, dx)$, with a discrete spectrum. Its lowest eigenvalue is 0 and a corresponding eigenvector is given by 1.

7.3. Existence of the Limit Correlation

We want to treat the important case where $c(x) = x_1 \cdot x_j$. We prove here a part of Theorem 1.5 which assumes more information on the structure of the family of functions $\Phi^{(m)}$:

$$\Psi^{(m)}(x) = \beta \sum_{i=1}^{m} W(x_i, x_{i+1}) \tag{7.10}$$

Proposition 7.3. Under the preceding assumptions and if $|\beta|$ is small enough, then the correlation

$$\mathrm{Cor}^{(m)}(i, j) = \langle x_i x_j \rangle_m - \langle x_i \rangle_m \langle x_j \rangle_m$$

admits, for all pair (i, j) s.t. $d(i, j) \geqslant 2$, a limit as $m \to \infty$.

We first explain here how we can arrive at the assumptions of Section 7.1. We have already observed that $\langle x_i \rangle_m$ and $\langle x_j \rangle_m$ have a limit. (If we assume that Ψ is even, we have $\langle x_j \rangle_m = \langle x_i \rangle_m = 0$.) A first integration by parts in the computation of $\langle x_i x_j \rangle$ gives

$$\langle x_i x_j \rangle = \langle x_i \cdot \partial_j \Psi \rangle$$

We can work directly with this new expression, or perform a new integration by parts:

$$\langle x_i x_j \rangle = \langle -\partial_{ij}^2 \Psi \rangle + \langle \partial_i \Psi \cdot \partial_j \Psi \rangle \tag{7.11}$$

In the case where (7.10) is satisfied, we get for $d(i, j) \geqslant 2$

$$\langle x_i x_j \rangle = \langle \partial_i \Psi \cdot \partial_j \Psi \rangle \tag{7.12}$$

which depends only on the variables (x_i, x_{i+1}, x_{i-1}) and (x_j, x_{j+1}, x_{j-1}) and is independent of m for m large. The assumption on Hess W is made in order to verify (1.23) and (1.29). All the other assumptions are easy to verify and we can apply Proposition 1.7. If we want now to prove the estimate on the correlations, we just observe that, if v_1 is the solution of the basic equation corresponding to $c(x) = x_1$, then the correlation is given by $\langle (v_1)_j \rangle_m$ by a simple argument of integration by parts. Then using the weight $\rho^m(j) = \exp[\kappa d(j, 1)]$, we obtain, in the same way essentially that we arrive at (7.6), the estimate

$$\sup_x |(v_1)_j (x)| \leqslant C \exp -\kappa d(j, 1) \quad \blacksquare \tag{7.13}$$

We shall come back to this estimate and its improvement in Sections 10 and 11.

Remark 7.4. We have implicitly assumed in the proof that $\nabla W(0) = 0$. This assumption can be eliminated using Remark 7.2.

7.4. Extension to More General Interactions

In order to get analogous results for the Schrödinger operator in Part III, we shall need a more general result if we make more general assumptions on the Φ's which no longer have a "finite-range" interaction. Inspection of the proof of Propositions 7.3 and 1.7 gives that Proposition 7.3 is also true under the weaker assumptions given in Theorem 1.6 and the following additional condition:

$$\|\nabla^4(\Psi^{(m+n)} - (\Psi^{(m)} \oplus \Psi^{(n)}))\|_{\mathscr{L}(B_1 \times B_1 \times B_1, B)} \leqslant C \qquad \text{for any } B \text{ in } \mathscr{B}_2 \tag{7.14}$$

We now have to analyze for fixed i and j two terms $\langle -\partial_{ij}^2 \Psi \rangle$ and $\langle \partial_i \Psi \cdot \partial_j \Psi \rangle$, which no longer depend on a finite number of variables.

The First Term. The estimate corresponds to different steps:

Step a. We remark that

$$\langle \partial_{ij}^2 \Psi^{(m)} \rangle_m = \langle \partial_{(i+l),(j+l)}^2 \Psi^{(m)} \rangle_m$$

and we shall take $l = [m/2]$.

Step b. We observe that $(\partial^2_{(i+l),(j+l)}\Psi^{(m)})(x^{(m)})$ is exponentially convergent. This is a consequence of (1.31), which gives for any x in $\mathbb{R}^{(m+n)}$

$$|(\partial^2_{(i+l),(j+l)}\Psi^{(m+n)})(x)-(\partial^2_{(i+l),(j+l)}\Psi^{(m)})(x')\leqslant C\exp-(\kappa m/2)$$

and of the exponential convergence of $x_k^{(m)}$ to y for any k given by Proposition 6.2. We have indeed

$$|\partial^2_{(i+l),(j+l)}\Psi^{(m)}(x^m)-\partial^2_{(i+l),(j+l)}\Psi^{(m)}(y,..., y)|\leqslant Cm\,|y-x_1^{(m)}|$$

using the invariance by permutation and (6.25).

Step c. For $l=[m/2]$ we observe that $|\nabla\partial^2_{(i+l),(j+l)}\Psi^{(m)}|$ is bounded in $l^\infty_{\rho^{(l,m)}}$ with $\rho^{(l,m)}(p)$ satisfying

$$\rho^{(l,m)}(q)=1 \qquad \text{for} \quad |q-l|\leqslant m/8$$

$$\rho^{(l,m)}(q)=\exp\kappa m/8 \qquad \text{for} \quad |q-l|\geqslant m/2$$

This is a consequence of the assumption (6.25).

In the same way, using the assumption (6.26), we obtain that $|\nabla^{(2)}\partial^2_{(i+l),(j+l)}\Psi^{(m)}|$ is bounded in $\mathscr{L}(l^\infty, l^\infty_{\rho^{(l,m)}})$, and by (the proof of) Proposition 5.2, we consequently get the control of Hess $f^{(m+n,l)}$ given by (7.6) with $l=[m/2]$, if we have the control of the family of functions

$$g^{(m,n)}(x,s)=s(\partial^2_{(i+l),(j+l)}\Psi^{(m+n)})+(1-s)(\partial^2_{(i+l),(j+l)}\Psi^{(m)})$$

When we solve the basic equation with

$$w^{(m,n)}(x,s)=\nabla g^{(m,n)}(x,s), \qquad \Phi=\Phi^{(m,n)}(x,s)$$

we have to apply a parameter-dependent version of Proposition 5.2 with $\rho=1$ and $\hat\rho=\rho_{m,n}$. We observe that

$$\begin{aligned}\partial_s\nabla g^{(m,n)}(x,s)&=(\partial^2_{(i+l),(j+l)}\nabla\Psi^{(m+n)})-((\partial^2_{(i+l),(j+l)}\nabla\Psi^{(m)})\\&=\partial^2_{(i+l),(j+l)}\nabla(\Psi^{(m+n)}(x)-[\Psi^{(m)}(x')+\Psi^{(n)}(x'')])\end{aligned}$$

Here we observe that the assumption (6.27) gives the condition that $\partial_s\nabla g^{(m,n)}(x,s)$ is bounded in $l^\infty_{\hat\rho}$, as needed in order to obtain (7.7). Now we observe that the control of $\partial_s\nabla^2 g^{(m,n)}(x,s)$ is given using the assumption (7.14), which implies the control of $\partial_s\nabla^2 g^{(m,n)}(x,s)$ in $\mathscr{L}(l^\infty, l^\infty_{\rho_{mn}})$.

The Second Term $\langle\partial_i\Psi\cdot\partial_j\Psi\rangle$. Following the proof which was made for the first term, we observe that

$$\langle\partial_i\Psi^{(m)}\cdot\partial_j\Psi^{(m)}\rangle=\langle\partial_{i+l}\Psi^{(m)}\cdot\partial_{j+l}\Psi^{(m)}\rangle$$

and we take again $l=[m/2]$.

We prove next the exponential convergence of the expression

$$\nabla(\partial_{i+l}\Psi^{(m)}\cdot\partial_{j+l}\Psi^{(m)})(x^{(m)})$$

For the third step, we observe, using the assumptions (1.9) and (1.29), that $|\nabla(\partial_{i+l}\Psi^{(m)}\cdot\partial_{j+l}\Psi^{(m)})|$ is bounded in $l^{\infty}_{\rho(l,m)}$. To end the proof, we observe the following properties:

$\nabla g^{(m,n)}(x,s)$ is in l^{∞} with $g^{(m,n)}(x,s)=\partial_{i+l}\Psi^{(m,n)}\cdot\partial_{j+l}\Psi^{(m,n)}$ (already observed).

$\nabla^2 g^{(m,n)}(x,s)$ is in $\mathscr{L}(l^{\infty})$ [using (1.28)–(6.25)].

$\partial_s\nabla g^{(m,n)}(x,s)$ is in $l^{\infty}_{\rho_{mn}}$ [using (1.28)–(1.31)].

$\partial_s\nabla^2 g^{(m,n)}(x,s)$ is in $\mathscr{L}(l^{\infty}, l^{\infty}_{\rho_{mn}})$ [using (1.28)–(6.27)].

This permits us to conclude the proof of the statements given at the beginning of this subsection. ∎

In the same way, we can probably treat more general correlations (see Section 13), but we shall have to use more explicitly the theory of the 0-standard functions. We emphasize that for the moment we are just analyzing the convergence of these correlations. It appears to be a problem which is partially independent of the problem of obtaining bounds on the correlations. This problem will be analyzed in Part II.

7.5. Existence of the Limit Measure

In this subsection we shall briefly describe how we can as a consequence of Proposition 1.7 deduce the existence of a "limit" measure.

Proposition 7.5. Under the assumptions of Proposition 1.7, there exists a unique measure $\mu^{(\infty)}$ on $\mathbb{R}^{\mathbb{N}}$ such that by introducing the projections π_k from $\mathbb{R}^{\mathbb{N}}$ onto $\mathbb{R}^k$: $x\to(x_1,\dots,x_k)$, and, for $k\leqslant m$, $\pi_{k,m}$ from $\mathbb{R}^m$ onto $\mathbb{R}^k$: $x^{(m)}\to(x_1,\dots,x_k)$, we have, for all c in $C_0^{\infty}(\mathbb{R}^k)$,

$$\lim_{m\to\infty}\langle c^{(m)}\rangle_m=\int_{\mathbb{R}^k}c\cdot d\mu_k^{(\infty)} \tag{7.15}$$

with

$$\mu_k^{(\infty)}=\mu^{(\infty)}\circ(\pi_k)^{-1}$$

Remark 7.6. One can reformulate (7.15) by saying that, for any k, the family of probability measures $\mu_k^m=\mu^{(m)}\circ(\pi_{k,m})^{-1}$ converges as m tends to ∞ in the weak sense to $\mu_k^{(\infty)}$.

This type of problem occurs of course quite naturally in statistical mechanics and in field theory and we refer to the book of Glimm and Jaffe[(9)] for an introduction. At the level of probability theory our reference is, as suggested in the same book, the book by Billingsley.[(2)]

Proof of the Proposition. By use of Kolmogorov's Theorem, and according to the compatibility conditions satisfied by the measures μ_k^m

$$\mu_{k+1}^m \circ \pi_{(k,k+1)}^{-1} = \mu_k^m \qquad \text{for} \quad m \geqslant k+1$$

we can work for fixed k and prove the existence of a limit measure μ_k^∞ on $\mathbb{R}^k$. The only problem is to prove that

$$f \to \lim_{m \to \infty} \int f \, d\mu_k^m$$

defines a probability measure. One way is to prove that the family $(\mu_k^m)_{m \in \mathbb{N}, m \geqslant k}$ is relatively compact (for the weak convergence topology) and Prohorov's Theorem says that it is sufficient to prove that the family is tight, that is: For any $\varepsilon > 0$ there exists a compact K in $\mathbb{R}^k$ such that

$$\mu_k^m(K) \geqslant 1 - \varepsilon; \qquad \forall m \geqslant k$$

In order to verify this last point, we observe that if we take $K = B(0, L) \subset \mathbb{R}^k$, there exists a constant C_k, independent of m, such that

$$\mu_k^m(\complement K) \leqslant C_k / L^2$$

For this we observe that (we take $h = 1$)

$$\begin{aligned}
\mu_k^m(\complement K) &= \int_{\complement K \times \mathbb{R}^m} \exp[-\Phi^{(m)}(x)] \, dx \Big/ \int_{\mathbb{R}^m} \exp[-\Phi^{(m)}(x)] \, dx \\
&\leqslant \frac{1}{L^2} \int_{\complement K \times \mathbb{R}^m} \left(\sum_{l=1}^k |x_l|^2 \right) \exp[-\Phi^{(m)}(x)] \, dx \Big/ \int_{\mathbb{R}^m} \exp[-\Phi^{(m)}(x)] \, dx \\
&\leqslant \frac{1}{L^2} \int_{\mathbb{R}^m} \left(\sum_{l=1}^k |x_l|^2 \right) \exp[-\Phi^{(m)}(x)] \, dx \Big/ \int_{\mathbb{R}^m} \exp[-\Phi^{(m)}(x)] \, dx
\end{aligned}$$

But the last sum,

$$\int_{\mathbb{R}^m} \left(\sum_{l=1}^k |x_l|^2 \right) \exp[-\Phi^{(m)}(x)] \, dx \Big/ \int_{\mathbb{R}^m} \exp[-\Phi^{(m)}(x)] \, dx$$

is bounded independently of m (make, for example, the integration by parts as in Section 7.1). ∎

7.6. Existence of the Magnetization, Regularity

Let us consider the following family of potentials:

$$\Phi^{(m)}(x, B) = \tfrac{1}{2} \sum_{l} (x_l - B)^2 + \Psi^{(m)}(x) \tag{7.16}$$

Then (inspired by connected problems given, for example, in ref. 18) we define the magnetization by

$$\mathcal{M}(m, B) = (1/m)(\partial \langle 1 \rangle_m / \partial B) / \langle 1 \rangle_m \tag{7.17}$$

Let us now prove Theorem 1.8.

$\mathcal{M}(m, B)$ is convergent (exponentially rapidly) to a continuous limit. In particular, if $\Psi^{(m)}$ is even, then the limit as B tends to 0 is zero. We observe that

$$\mathcal{M}(m, B) = \langle (x_1 - B) \rangle_m \tag{7.18}$$

It is then immediate to apply Proposition 1.7, observing that all the assumptions are uniformly satisfied with respect to B if $\|W\|$ is small enough. One point, however, is different because the critical point $x^{(m)}(B)$ is moving with B. But we have seen how to control this problem in Proposition 6.2. As a consequence, $x_1^{(m)}(B)$ is exponentially convergent and this permits us to solve the difficulty. Because all the proofs are based on the Cauchy criterion, it is clear that the limit is continuous with respect to B. ▮

Remark 7.7. Using Proposition 5.2 and adding in all the proofs the parameter B in the parameter-dependent class of 0-standard functions, we can probably obtain the C^∞ dependence with respect to B. In particular the C^1 control will permit us to study the susceptibility which corresponds to the derivative with respect to B of the magnetization.

Remark 7.8. As in the case of the correlation, we can also analyze the more general interactions introduced in Theorem 1.6 and without the condition (1.21).

PART II. PRECISE ESTIMATES FOR THE CORRELATIONS

8. SEMICLASSICAL COMPUTATIONS

Assume $\Phi(x)$ convex and even. We are interested in

$$\frac{\int x_j x_k \exp[-\Phi(x)/h]\, dx}{\int \exp[-\Phi(x)/h]\, dx} \tag{8.1}$$

This quantity is equal to $c_{j,k}(h)$, where we solve

$$\begin{aligned} &x_j x_k \exp[-\Phi(x)/h]\, dx \\ &\quad = c_{j,k}(h) \exp[-\Phi(x)/h]\, dx + h\mathscr{L}_v(\exp[-\Phi(x)/h]\, dx) \end{aligned} \tag{8.2}$$

or more explicitly

$$x_j x_k = c_{j,k} - v(x, \partial_x; h)(\Phi) + h \operatorname{div} v \tag{8.3}$$

We then try formal expansions of the type

$$c_{j,k}(h) \sim c_{j,k}^1 h + c_{j,k}^2 h^2 + \cdots$$
$$v \sim v^0 + hv^1 + \cdots$$

(vector fields in x).

This gives the equations

$$\begin{aligned} x_j x_k &= -\langle v^0(x, \partial_x), \nabla\Phi\rangle \\ c_{j,k}^1 &+ \operatorname{div} v_0(0) = 0 \end{aligned} \tag{8.4}$$

Taking $v^0 = \nabla f^0$, we get

$$x_j x_k = \nabla\Phi \cdot \partial_x f^0, \qquad c_{j,k}^1 = -\Delta f^0(0) \tag{8.5}$$

so f^0 vanishes to the second order at 0. The first equation of (8.5) implies for the Hessians

$$x_j x_k = \langle (\text{Hess}\,\Phi(0))x, ((\text{Hess}\, f^0)))x\rangle \tag{8.6}$$

Let $\varepsilon_{j,k}$ be the $m \times m$ matrix:

$$(\varepsilon_{j,k})_{pq} = \delta_{pj} \cdot \delta_{qk}$$

Then (8.6) means that

$$\varepsilon_{j,k} + \varepsilon_{k,j} = (\text{Hess}\,\Phi(0)) \circ (\text{Hess}\, f^0(0)) + (\text{Hess}\, f^0(0)) \circ (\text{Hess}\,\Phi(0)) \tag{8.7}$$

which can be rewritten as

$$e^{t(\text{Hess}\,\Phi(0))}(\varepsilon_{j,k} + \varepsilon_{k,j})e^{t(\text{Hess}\,\Phi(0))} = \frac{\partial}{\partial t}\left(e^{t(\text{Hess}\,\Phi(0))}(\text{Hess}\, f^0(0))e^{t(\text{Hess}\,\Phi(0))}\right)$$

so

$$(\text{Hess}\, f^0(0)) = \int_{-\infty}^0 e^{t(\text{Hess}\,\Phi(0))}(\varepsilon_{j,k} + \varepsilon_{k,j})e^{t(\text{Hess}\,\Phi(0))}\, dt \tag{8.8}$$

(This would also follow from taking Hessians of the direct solution of (8.5):

$$f^0(x) = \int_{-\infty}^0 (x_j x_k)[\exp(t\,\nabla\Phi \cdot \partial_x)(x)]\, dt$$

Here

$$\Delta f^0(0) = \mathrm{tr}(\mathrm{Hess}\, f^0(0)) = \mathrm{tr}\, 2 \int_{-\infty}^{0} e^{t(\mathrm{Hess}\, \Phi(0))} \varepsilon_{j,k} e^{t(\mathrm{Hess}\, \Phi(0))}\, dt$$

$$= \mathrm{tr}\, 2 \int_{-\infty}^{0} \varepsilon_{j,k} e^{2t(\mathrm{Hess}\, \Phi(0))}\, dt = \mathrm{tr}((\mathrm{Hess}\, \Phi(0))^{-1}\, \varepsilon_{j,k})$$

$$= ((\mathrm{Hess}\, \Phi(0))^{-1})_{jk} \tag{8.9}$$

We can now formulate the *difficulty*: Under reasonable assumptions $((\mathrm{Hess}\, \Phi(0))^{-1})_{jk}$ will be exponentially decaying when $|j-k| \to \infty$, and in order to get a satisfactory asymptotic result valid uniformly w.r.t. h when $h > 0$ is sufficiently small, we would like (at least) to control the whole procedure with exponential weights which decay (or increase) with the same rate as the decay in $((\mathrm{Hess}\, \Phi(0))^{-1})_{jk}$. It would therefore *not* be sufficient to have estimates with weights ρ for which $((\mathrm{Hess}\, \Phi(0))^{-1})$ is uniformly bounded from l^p_ρ into itself, as can be understood if we think of the case when $((\mathrm{Hess}\, \Phi(0))^{-1})$ is something like $\exp -|j-k|$. In this case the norm in $\mathscr{L}(l^p_\rho)$, now with $\rho = \rho_\varepsilon(j) = \exp(1-\varepsilon) j$, is $\mathcal{O}(1/\varepsilon)$. The idea is then that, in order to control the element at (j, k) of $(\mathrm{Hess}\, \Phi(0))^{-1}$, it is enough to control the $l^1_\rho \to l^\infty_\rho$ for suitable weights, and for such weaker estimates we can hope for a larger class of weights. [If we think about the example $\exp -|j-k|$, we see that this norm is $\mathcal{O}(1)$ for $\rho(j) = \exp j$, so we have then avoided the ε loss.] We shall see in the next section how to get efficient $l^1_\rho \to l^\infty_\rho$ bounds for $(\mathrm{Hess}\, \Phi(0))^{-1}$. First we end this section by doing what we did in Section 7.3. We first solve

$$x_k \exp[-\Phi(x)/h]\, dx$$
$$= c_k(h) \exp[-\Phi(x)/h]\, dx + h\mathscr{L}_{v_k}(\exp[-\Phi(x)/h]\, dx) \tag{8.10}$$

(and here we observe that $c_k = \langle x_k \rangle = 0$, since we assumed Φ to be even). Then, as in Section 7.3, we get

$$\int x_j x_k \exp[-\Phi(x)/h]\, dx$$

$$= \left(c_k \int x_j \exp[-\Phi(x)/h]\, dx\right) + h\left(\int x_j \mathscr{L}_{v_k}(\exp[-\Phi(x)/h]\, dx)\right)$$

$$= c_k c_j \int \exp[-\Phi(x)/h]\, dx + h \int (\mathscr{L}_{x_j v_k} - \langle dx_j, v_k \rangle) \exp[-\Phi(x)/h]\, dx$$

$$= c_k c_j \int \exp[-\Phi(x)/h]\, dx - h \int v_{k,j}(x; h) \exp[-\Phi(x)/h]\, dx \tag{8.11}$$

where $v_k(x, \partial_x; h) = \sum_j v_{k,j} \partial_{x_j}$.

Using that $c_j c_k = 0$, we get the h-asymptotic expansion of the quantity (8.1):

$$c_{j,k}(h) \sim -hv_{k,j}^0(0) + \mathcal{O}_{j,k}(h^2) + \cdots$$

The absolute value of (8.1) is then smaller than or equal to $h \|v_{k,j}\|_{L^\infty}$. Writing $x_k = -\langle v_k^0, \nabla\Phi\rangle$, differentiating with respect to x, and putting $x=0$, we get

$$\delta_{k,j} = -\sum_{\nu=1}^{m} v_{k,\nu}^0(0)\, \partial_{x_\nu}\partial_{x_j}\Phi(0)$$

or

$$e_k = -(\text{Hess}\ \Phi(0))\, v_k^0(0)$$

(where e_k is the kth unit vector).

Hence

$$v_k^0(0) = -(\text{Hess}\ \Phi(0))^{-1}\, e_k$$

and $-v_{k,j}^0(0)$ is (again of course) equal to $((\text{Hess}\ \Phi(0))^{-1})_{j,k}$.

9. WEIGHTED $l^1 \to l^\infty$ ESTIMATES FOR INVERSE MATRICES

Consider first the situation as it will appear after conjugation with a weight. Let D be a diagonal matrix $\geqslant r_0 > 0$. Let A be a matrix (not necessarily symmetric) with $\|A\|_{\mathcal{L}(l^\infty)} < r_0$. Let $x \in \mathbb{R}^m$ and assume for some $j \in \{1,..., m\}$ that $x_j = |x|$ and that $((D+A)x)_j = 0$, i.e., $d_j x_j + (Ax)_j = 0$. Then $x_j = -(1/d_j)(Ax)_j$ and we get $|x| = x_j \leqslant (\|A\|/d_j)\,|x|$ and $\|A\|/d_j < 1$, so necessarily $x = 0$. Hence we get, under the assumptions D diagonal $\geqslant r_0 > 0$ and $\|A\|_{\mathcal{L}(l^\infty)} < r_0$, the inequality

$$|x|_\infty \leqslant \max_{(j;\,((D+A)x)_j \neq 0)} |x_j| \tag{9.1}$$

We now slightly change the assumptions and let $D+A$ be the unconjugated matrix corresponding to Hess $\Phi(x)$. We still assume that D is diagonal $\geqslant r_0 > 0$ and we assume that $(D+A)^{-1}$ exists and satisfies

$$\|(D+A)^{-1}\|_{\mathcal{L}(l^\infty)} \leqslant C_0 \tag{9.2}$$

Let $\rho = \rho(j) > 0$ and assume that

$$\|\rho A \rho^{-1}\|_{\mathcal{L}(l^\infty)} < r_0$$

Consider the equation

$$(D+A)x=y$$

which can be rewritten as

$$(D+\rho A\rho^{-1})\,\rho x=\rho y$$

We can then apply the earlier discussion and obtain

$$|\rho x|_\infty \leqslant \max_{(j:\,y_j\neq 0)} |(\rho x)_j| \tag{9.3}$$

Assuming that $\rho(j)=1$ whenever $y_j\neq 0$ (so the choice of ρ will depend on y), we get from (9.2) and (9.3)

$$|\rho x|_\infty \leqslant C_0\,|y|_\infty \tag{9.4}$$

The nice thing with these estimates is that we do not have to introduce any factor $1/\varepsilon$, with $\varepsilon=r_0-\|\rho A\rho^{-1}\|_{\mathscr{L}(l^\infty)}$, into the r.h.s.

10. NEW ESTIMATES FOR THE VECTOR-FIELD EQUATION

We neglect in the application of the maximum principle the problem at ∞, which can be solved as in Section 5. Equations (8.3) and (8.10) are of the type

$$g(x)=c-v(x,\partial_x;h)(\Phi)+h\operatorname{div} v$$

and if we look for v of the type ∇f, we get

$$g(x)=c-\nabla\Phi\cdot\partial_x f+h\,\Delta f$$

Taking the gradient, we get

$$w(x)=-(\nabla\Phi)(x)\cdot(\partial_x v)(x)+h(\Delta v)(x)-\text{Hess}\,\Phi(x)\,v(x)$$

with $w=\nabla g$, $v=\nabla f$.

We assume that $v(x)$ tends to 0 as $|x|\to\infty$. We also assume that

$$\text{Hess}\,\Phi(x)=D(x)+A(x) \tag{10.1}$$

with $D(x)$ diagonal and $\geqslant r_0>0$,

$$\|\rho A(x)\rho^{-1}\|_{\mathscr{L}(l^\infty)}<r_0 \tag{10.2}$$

for some weight $\rho=\rho(j)>0$, independent of x. Let us also assume

$$\|\text{Hess}\,\Phi(x)^{-1}\|_{\mathscr{L}(l^\infty)}<C_0,\qquad \forall x\in\mathbb{R}^m \tag{10.3}$$

Let $x_0 \in \mathbb{R}^m$ be a point where $\sup_x |\rho v(x)|$ is attained and denote this supremum by M. Let $j \in \{1,..., m\}$ have the property that $|\rho(j) v_j(x_0)| = M$. Then $(\partial_x v_j)(x_0) = 0$ and $v_j(x_0) \cdot (\Delta v_j)(x_0) \leqslant 0$. Hence, if we rewrite (10.2) as

$$\rho w = -(\nabla\Phi) \cdot (\partial_x \rho v) + h(\Delta \rho v) - (\rho \text{ Hess } \Phi \rho^{-1})(\rho v) \tag{10.4}$$

we get [assuming in order to fix the ideas that $v_j(x_0) > 0$] that

$$\rho(j)\, w_j(x_0) \leqslant -((\rho \text{ Hess } \Phi(x_0)\, \rho^{-1}(\rho v(x_0)))_j$$

Adding the assumption that $w_j(x_0) = 0$ (for the j in consideration), we get

$$D_j(x_0)\, \rho v_j(x_0) + ((\rho A(x_0) \rho^{-1})(\rho v(x_0))_j \leqslant 0$$

and, as in Section 9, we get a contradiction unless $M = 0$. We have then showed that

$$\sup_{x \in \mathbb{R}^m} |\rho v(x)| \leqslant \max_{(j, w_j(x_0)) \neq 0} |\rho(j)\, v_j(x_0)| \tag{10.5}$$

We apply this to (8.10). Then w is the kth unit vector and if we assume (after renormalization of ρ) that $\rho(k) = 1$, we get from (10.3), (10.5)

$$\sup_{x \in \mathbb{R}^m} |\rho v(x)| \leqslant C_0 \tag{10.6}$$

In other words,

$$|v_{k,j}(x)| \leqslant C_0(1/\rho(j))$$

provided that ρ has all the required properties and that $\rho(k) = 1$. Combining this with (8.10), we get

$$\frac{|\int x_j x_k \exp[-\Phi(x)/h]\, dx|}{\int \exp[-\Phi(x)/h]\, dx} \leqslant \frac{C_0 h}{\rho(j)} \tag{10.7}$$

We shall present in the next section an example coming from statistical mechanics where these assumptions are satisfied.

11. AN EXAMPLE

As already mentioned in the introduction, this potential appears in a course by Kac.[18] The estimates we give here are reminiscent of ref. 27. Let $\nu \in]0, 1/4[$ be fixed, and consider

$$\Phi(x) = \tfrac{1}{2} \sum x_j^2 - 2 \sum \ln \cosh[(\nu/2)^{1/2} (x_j + x_{j+1})]$$

with $j \in \mathbb{Z}/m\mathbb{Z}$. Then[27] Hess $\Phi = I + A(x)$ with

$$A(x) = \begin{pmatrix} d_1(x) & c_1(x) & 0 & \cdot & \cdot & 0 & c_m(x) \\ c_1(x) & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & c_{m-1}(x) \\ c_m(x) & 0 & \cdot & \cdot & 0 & c_{m-1}(x) & d_m(x) \end{pmatrix}$$

with $|d_j(x)| \leqslant 2\nu$, $|c_j(x)| \leqslant \nu$, $d_j(0) = -2\nu$, $c_j(0) = -\nu$.

Then, if $\rho(j) > 0$, we have

$$\begin{aligned} \|\rho A(x) \rho^{-1}\|_{\mathscr{L}(l^\infty)} &\leqslant \sup_j \nu(2 + [\rho(j)/\rho(j+1)] + [\rho(j)/\rho(j-1)]) \\ &= 2\nu(1 + \sup_j [\mu(j) + 1/\mu(j+1)]/2) \end{aligned} \tag{11.1}$$

with

$$\mu(j) \stackrel{\text{def}}{=} (\rho(j)/\rho(j-1)) \stackrel{\text{def}}{=} \exp(\omega(j)) \tag{11.2}$$

[so $\sum_j \omega(j) = 0$, since we are in the cyclic case].

According to the earlier discussion, we are interested in choices of ρ with

$$2\nu(1 + \sup_j [\mu(j) + 1/\mu(j+1)]/2) < 1$$

or equivalently

$$\sup_j [\mu(j) + 1/\mu(j+1)]/2 < (1 - 2\nu)/2\nu \tag{11.3}$$

Here we shall take ω to be a continuous m-periodic function which is linear on $[j, j+1]$ for every j and which satisfies

$$\cosh \omega(j) < (1 - 2\nu)/2\nu; \qquad \sum_0^{m-1} \omega(j) = 0 \tag{11.4}$$

The question is then to find some further sufficient condition which implies (11.3). We have

$$\begin{aligned}
&\tfrac{1}{2}[\exp \omega(j)+\exp -\omega(j+1)]\\
&\quad=\cosh \omega(j)+\tfrac{1}{2}[\exp -\omega(j+1)-\exp -\omega(j)]\\
&\quad\leqslant \cosh \omega(j)+\sup(\exp -\omega(j+1), \exp -\omega(j))\,|\omega'(j+\tfrac{1}{2})|\\
&\quad\leqslant \cosh \omega(j)+\tfrac{1}{2}\sup(\cosh \omega(j+1), \cosh \omega(j))\,|\omega'(j+\tfrac{1}{2})|\\
&\quad\leqslant \cosh \omega(j)+[(1-2v)/2v]\,|\omega'(j+\tfrac{1}{2})| \qquad (11.5)
\end{aligned}$$

Let $\omega_0>0$ be the solution of $\cosh \omega_0=(1-2v)/2v$. In order to have (11.3) it then suffices to have (11.4) and

$$[(1-2v)/2v]\,|\omega'(j+\tfrac{1}{2})|<\cosh \omega_0-\cosh \omega(j), \qquad \forall j$$

or equivalently,

$$|\omega'(j+\tfrac{1}{2})|\leqslant[\cosh \omega_0-\cosh \omega(j)]/\cosh(\omega_0), \qquad \forall j \qquad (11.6)$$

We have

$$\begin{aligned}
&[\cosh \omega_0-\cosh \omega(j)]/\cosh(\omega_0)\\
&\quad=\{[\omega_0-\omega(j)]/\cosh \omega_0\}\left\{\int_{\omega(j)}^{\omega_0}\sinh t\,dt/[\omega_0-\omega(j)]\right\}
\end{aligned}$$

If we assume for some j that $\omega(j)\geqslant 0$, we get, since $\sinh t$ is increasing on $[0,+\infty]$,

$$\begin{aligned}
&[\cosh \omega_0-\cosh \omega(j)]/\cosh \omega_0\\
&\quad\geqslant\{[\omega_0-\omega(j)]/\cosh \omega_0\}\int_0^{\omega_0}\sinh t\,dt/\omega_0\\
&\quad=\{[\omega_0-\omega(j)]/\cosh \omega_0\}(\cosh \omega_0-1)/\omega_0=[\omega_0-\omega(j)]/C(v)
\end{aligned}$$

with

$$\begin{aligned}
C(v)&=\omega_0\cosh \omega_0/(\cosh \omega_0-1)\\
&=(1-2v)/(1-4v)\cdot\cosh^{-1}[(1-2v)/2v] \qquad (11.7)
\end{aligned}$$

A similar discussion holds in the case $\omega(j)\leqslant 0$, so in order to have (11.3), it suffices to have (11.4) and

$$|\omega'(j+\tfrac{1}{2})|\leqslant\min[(\omega_0-\omega(j)], [\omega_0+\omega(j)])/C(v) \qquad (11.8)$$

(Notice that (11.4) implies that $\omega(j)\in\,]-\omega_0,\omega_0[$).

Roughly we are then allowed to take ω approaching ω_0 or $-\omega_0$ exponentially fast. More precisely, if $\tilde{C}(\nu)>0$ is sufficiently large, we can take ω piecewise linear with

$$\begin{aligned} &\omega(j)=\omega_0(1-\exp[-j/\tilde{C}(\nu)]), && j\in[0,m/4]\cap\mathbb{Z} \\ &\omega(j)=\omega((m/2)-j), && j\in[m/4,m/2]\cap\mathbb{Z} \\ &\omega(j)=-\omega(-j), && j\in[-(m/2),0]\cap\mathbb{Z} \end{aligned} \tag{11.9}$$

For this choice of ω we get

$$\rho(j)=\exp[\omega_0\,|j|+\mathcal{O}(1)], \qquad |j|\leqslant m/2$$

Finally, returning to (10.7), we obtain

$$\frac{|\int x_0 x_j \exp-[\Phi(x)/h]\,dx|}{\int \exp-[\Phi(x)/h]\,dx}\leqslant C_1(\nu)\,h\exp(-\omega_0\,|j|), \qquad |j|\leqslant m/2 \tag{11.10}$$

The constant ω_0 seems to be optimal. It might be of interest to push this even further and get some explicit $C_1(\nu)$.

12. FKG INEQUALITIES WITH THE MAXIMUM PRINCIPLE

In this section we shall discuss with a new approach the celebrated FKG inequalities.[8] The following proposition is the corresponding analog of these inequalities in our context and is due to Cartier[6] (we refer also to refs. 21, 22, and 9 for interesting discussions or presentations of the material):

Proposition 12.1. If Φ is of class C^2 on $\mathbb{R}^m$ and such that $|\Phi(x)|\geqslant(|x|^k/C-C)$, for some k, $C>0$, and if

$$\partial^2\Phi/\partial x_i\,\partial x_j\leqslant 0 \qquad \text{for} \quad i\neq j \tag{12.1}$$

then we have the following inequality for the correlations attached to the measure $\exp-\Phi\,dx/(\int\exp-\Phi\,dx)$:

$$\langle x_i x_j\rangle_\Phi-\langle x_i\rangle_\Phi\,\langle x_j\rangle_\Phi\geqslant 0 \tag{12.2}$$

The result is true more generally if one replaces x_i, x_j by two functions $(g_1(x), g_2(x))$ on $\mathbb{R}^m$ which are monotone increasing with respect to each variable.

Proof of Proposition 12.1. (In the strictly convex case when the Hessian is uniformly bounded, using the maximum principle): Let f_1 be a solution of

$$g_1(x) = c_1 + \nabla\Phi\, \nabla f_1 - \Delta f_1$$

The quantity we want to compute is given by

$$\langle g_1 g_2 \rangle_\Phi - \langle g_1 \rangle_\Phi \langle g_2 \rangle_\Phi = \langle v_1 \cdot \nabla g_2 \rangle_\Phi$$

with $v_1 = \nabla f_1$.

It is consequently sufficient to prove that:

Lemma 12.2. Under the assumptions (2.2), (2.3), and if the coefficients of $\omega = \nabla g$ are positive and satisfy

$$|\partial_x^\alpha w| \leqslant C_\alpha, \qquad \forall \alpha \in \mathbb{N}^m$$

then the coefficients of $v = \nabla f$ solutions of Eq. (3.2) are positive.

Proof of Lemma. Let us start from the basic equation (3.2):

$$w = \nabla\Phi\, \nabla v - \Delta v + \text{Hess}\, \Phi v \tag{12.3}$$

We treat the case where w and Ψ are with compact support. We decompose v in the form $v = v^+ - v^-$ where $v_j^+ = \sup(0, v_j)$. We want to prove that v^- is equal to 0. Let x_0 be a point where $\|v^-(x)\|_{l^2}$ attains its maximum. We multiply Eq. (12.3) by $v^-(x_0)$ and we get

$$\langle w(x), v^-(x_0) \rangle = (\nabla\Phi\nabla - \Delta) \langle v, v^-(x_0) \rangle + \langle \text{Hess}\, \Phi v, v^-(x_0) \rangle \tag{12.4}$$

We observe now that $x \to (-\langle v, v^-(x_0) \rangle)$ has a maximum at x_0 and we get

$$\langle w(x_0), v^-(x_0) \rangle \leqslant \langle \text{Hess}\, \Phi v^+, v^- \rangle(x_0) - \langle \text{Hess}\, \Phi v^-, v^- \rangle(x_0)$$

and finally

$$\langle \text{Hess}\, \Phi v^-, v^- \rangle(x_0) \leqslant \langle \text{Hess}\, \Phi v^+, v^- \rangle(x_0)$$

We observe that the r.h.s. is negative and, if Hess $\Phi(x)$ is positive definite, we obtain $v^-(x_0) = 0$ and the lemma is proved with the restriction on the support on w and Φ. The argument of regularization is then analogous to the argument given in Section 5.

13. ESTIMATES FOR THE HIGHER-ORDER CORRELATIONS

Let Φ and $\mathscr{R}$ be as in Section 4. Let v_j be the vector fields determined by

$$\exp(-\Phi/h)(x_j - \langle x_j \rangle)\, dx = -h\mathscr{L}_{v_j}(\exp(-\Phi/h)\, dx), \qquad v_j = \nabla f_j \tag{13.1}$$

Here

$$\langle a \rangle = \frac{\int a(x) \exp(-\Phi/h)\, dx}{\int \exp(-\Phi/h)\, dx} \tag{13.2}$$

We now use that

$$\mathscr{L}_{av} = av \lrcorner d + d \circ av \lrcorner = a(v \lrcorner d + d \circ v \lrcorner) + da \wedge v \lrcorner = a\mathscr{L}_v + da \wedge v \lrcorner$$

which on forms of maximal degree becomes

$$\mathscr{L}_{av} = a\mathscr{L}_v + da \wedge v \lrcorner + v \lrcorner da \wedge = a\mathscr{L}_v + \langle v, da \rangle = a\mathscr{L}_v + v(a)$$

Hence

$$\begin{aligned} \int a(x)(x_j - \langle x_j \rangle) \exp(-\Phi/h)\, dx &= -h \int a(x)\, \mathscr{L}_{v_j}((\exp(-\Phi/h)\, dx) \\ &= -h \int [\mathscr{L}_{av_j} - v_j(a)][\exp(-\Phi/h)\, dx] \\ &= \int (hv_j a) \exp(-\Phi/h)\, dx \end{aligned} \tag{13.3}$$

or, in other words,

$$\langle (x_j - \langle x_j \rangle) a \rangle = \langle h v_j(a) \rangle$$

Let $j_1,..., j_k \in \{1,..., m\}$ with $j_\alpha \neq j_\beta$ if $\alpha \neq \beta$. (Here $\{j_1,..., j_k\}$ may vary with m, but k is assumed to be fixed.) We are interested in estimating the higher-order correlation that we define as

$$\langle x_{j_1},..., x_{j_k} \rangle_k = \langle (x_{j_1} - \langle x_{j_1} \rangle) \cdots (x_{j_k} - \langle x_{j_k} \rangle) \rangle \tag{13.4}$$

This definition appears for example in the article of Cartier.[6] Using (13.3) several times, we get $\langle x_{j_1},..., x_{j_k} \rangle_m$ is equal to a finite sum of terms of the form $h^{k-l} \langle v_{\mathscr{J}_1}(x) \cdots v_{\mathscr{J}_l}(x) \rangle$, where $\# \mathscr{J}_j \geqslant 2$, $\mathscr{J}_\alpha \cap \mathscr{J}_\beta = \varnothing$ for $\alpha \neq \beta$, $\{1,..., k\} = \mathscr{J}_1 \cup \cdots \cup \mathscr{J}_l$, and $\mathscr{J}_j$ are viewed as ordered sets: If $\mathscr{J} = (j_1,..., j_p)$, then

$$v_{\mathscr{J}}(x) \stackrel{\text{def}}{=} v_{j_1}(x, \partial_x) \cdots v_{j_{p-1}}(x, \partial_x)(x_{j_p})$$

Let $\hat{\rho}_j \in \mathscr{R}$ with $\hat{\rho}_j(j)=1$, and introduce the parameter $s_j \in [0, 1]$ associated with the weights $\hat{\rho}_j$. Then $s_j x_j$ is 0-standard in the parameter-dependent sense of Section 4. Then Proposition 4.1 gives the same result for $s_j f_j$. By induction over $k=2, 3, 4,...,$ we shall show:

(I_k) For $\# \mathscr{J} = k$, we have for all $l \geqslant 0$

$$\langle \nabla^l v_{\mathscr{J}}(x), t_1 \otimes \cdots \otimes t_l \rangle = \mathcal{O}_{k,l}(1) \prod_1^l |t_j|_{\infty, \rho_j}$$

when

$$1 \leqslant \left(\prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu \right) \left(\prod_1^l \rho_j \right), \qquad \tilde{\rho}_\nu \leqslant \hat{\rho}_\nu, \qquad ((\tilde{\rho}_\nu)_{\mathscr{J}}, \rho_1, ..., \rho_l) \in \mathscr{R}_{\# \mathscr{J} + l}$$

with the convention that for $l=0$ the estimate above reads

$$|v_{\mathscr{J}}(x)| = \mathcal{O}(1) \frac{1}{\inf_{n \in \{1,...,m\}} \prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu(n)}$$

Proof. In the case $k=2$, let $\mathscr{J} = \{j_1, j_2\}$, $v_{\mathscr{J}}(x) = \langle \nabla f_{j_1}, \nabla x_{j_2} \rangle$. Then

$$|v_{\mathscr{J}}(x)| \leqslant \frac{1}{\inf \hat{\rho}_{j_1} \hat{\rho}_{j_2}} |\nabla f_{j_1}|_{\infty, \hat{\rho}_{j_1}} \cdot |\nabla x_{j_2}|_{1, \hat{\rho}_{j_2}} = \mathcal{O}(1) \frac{1}{\inf \hat{\rho}_{j_1} \hat{\rho}_{j_2}}$$

For $l \geqslant 1$

$$\begin{aligned} t_1(\partial_x) \cdots t_l(\partial_x) \langle \nabla f_{j_1}, \nabla x_{j_2} \rangle &= \langle \nabla^{l+1} f_{j_1}, \nabla x_{j_2} \otimes t_1 \otimes \cdots \otimes t_l \rangle \\ &= \mathcal{O}(1) |\nabla x_{j_2}|_{1, \tilde{\rho}_{j_2}} \cdot \prod_1^l |t_j|_{\infty, \rho_j} = \mathcal{O}(1) \prod_1^l |t_j|_{\infty, \rho_j} \end{aligned}$$

if

$$1 \leqslant \tilde{\rho}_{j_1} \tilde{\rho}_{j_2} \left(\prod_1^l \rho_j \right), \qquad \tilde{\rho}_{j_1} \leqslant \hat{\rho}_{j_1}, \qquad \tilde{\rho}_{j_2} \leqslant \hat{\rho}_{j_2}, \qquad (\tilde{\rho}_{j_1}, \tilde{\rho}_{j_2}, \rho_1, ..., \rho_l) \in \mathscr{R}_{l+2}$$

We have then verified (I_2).

For some $k \geqslant 2$, we now assume that $(I_2),..., (I_k)$ hold, and we shall verify (I_{k+1}). Let $\tilde{\mathscr{J}} = (j_0, j_1, ..., j_k)$, $\mathscr{J} = (j_1, ..., j_k)$. Then

$$v_{\tilde{\mathscr{J}}} = v_{j_0}(x, \partial_x)\, v_{\mathscr{J}}(x) = \langle \nabla v_{\mathscr{J}}(x), \nabla f_{j_0}(x) \rangle$$

and using (I_k), we get

$$v_{\tilde{\mathscr{J}}}(x) = \mathcal{O}(1)\, |\nabla f_{j_0}(x)|_{\infty, \rho_1}$$

when

$$1 \leqslant \left(\prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu \right) \rho_1, \qquad \tilde{\rho}_\nu \leqslant \hat{\rho}_\nu, \qquad ((\tilde{\rho}_\nu)_{\mathscr{J}}, \rho_1) \in \mathscr{R}_{\# \mathscr{J}+1}$$

Take $\rho_1 = \tilde{\rho}_{j_0}$ where we assume that

$$\tilde{\rho}_{j_0} \leqslant \hat{\rho}_{j_0}, \qquad (\tilde{\rho}_{j_0}, (\tilde{\rho}_\nu)_{\mathscr{J}}) \in \mathscr{R}_{\# \mathscr{J}+1}$$

If $1 \leqslant (\prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu) \tilde{\rho}_{j_0}$, then

$$\langle \nabla v_{\mathscr{J}}(x), \nabla f_{j_0}(x) \rangle = \mathcal{O}(1) \, |\nabla f_{j_0}|_{\infty, \tilde{\rho}_{j_0}} = \mathcal{O}(1)$$

Replacing $\tilde{\rho}_{j_0}$ by $\tilde{\rho}_{j_0} / [\inf(\prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu) \cdot \tilde{\rho}_{j_0}]$, we get

$$\langle \nabla v_{\mathscr{J}}(x), \nabla f_{j_0}(x) \rangle = \mathcal{O}(1) \frac{1}{\inf(\tilde{\rho}_{j_0} \cdot \prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu)}$$

For $l \geqslant 1$, we have

$$\begin{aligned} & t_1(\partial_x) \cdots t_l(\partial_x) \langle \nabla v_{\mathscr{J}}, \nabla f_{j_0} \rangle \\ &= \sum_{\substack{A \cup B = \{1, \ldots, l\} \\ A \cap B = \varnothing}} \langle t_A(\partial_x) \nabla v_{\mathscr{J}}(x), t_B(\partial_x) \nabla f_{j_0} \rangle \\ &= \sum_{\substack{A \cup B = \{1, \ldots, l\} \\ A \cap B = \varnothing}} \langle \nabla^{1 + \# A} v_{\mathscr{J}}(x), t_A \otimes \langle \nabla^{\# B + 1} f_{j_0}, t_B \rangle \rangle \end{aligned} \tag{13.5}$$

where $\langle \nabla^{\# B + 1} f_{j_0}, t_B \rangle \rangle$ is defined by

$$\langle \langle \nabla^{\# B + 1} f_{j_0}, t_B \rangle \rangle, s \rangle = \langle \nabla^{\# B + 1} f_{j_0}, t_B \otimes s \rangle$$

Here the general term in the last sum is

$$\mathcal{O}(1) \prod_{j \in A} |t_j|_{\infty, \rho_j} \cdot |\langle \nabla^{\# B + 1} f_{j_0}, t_B \rangle|_{\infty, \tilde{\rho}} \tag{13.6}$$

if

$$1 \leqslant \prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu \cdot \prod_{j \in A} \rho_j \cdot \tilde{\rho}, \qquad \tilde{\rho}_\nu \leqslant \hat{\rho}_\nu, \qquad ((\tilde{\rho}_\nu)_{\nu \in \mathscr{J}}, (\rho_j)_{j \in A}, \tilde{\rho}) \in \mathscr{R}_{\# \mathscr{J} + \# A + 1} \tag{13.7}$$

Moreover,

$$|\langle \nabla^{\# B + 1} f_{j_0}, t_B \rangle|_{\infty, \tilde{\rho}} = \mathcal{O}(1) \prod_{j \in B} |t_j|_{\infty, \rho_j} \tag{13.8}$$

if

$$1 \leqslant \tilde{\rho}_{j_0} \cdot \frac{1}{\tilde{\rho}} \cdot \prod_{j \in B} \rho_j, \qquad \tilde{\rho}_{j_0} \leqslant \hat{\rho}_{j_0}, \qquad \left(\tilde{\rho}_{j_0}, \frac{1}{\tilde{\rho}}, (\rho_j)_{j \in B}\right) \in \mathscr{R}_{\# B+2} \quad (13.9)$$

Assume now that

$$\begin{aligned} &(\tilde{\rho}_{j_0}, (\tilde{\rho}_\nu)_{\nu \in \mathscr{J}}, \rho_1, ..., \rho_l) \in \mathscr{R}_{1+\#\mathscr{J}+l} \\ &1 \leqslant \tilde{\rho}_{j_0} \cdot \prod_{\nu \in \mathscr{J}} \tilde{\rho}_\nu, \qquad \tilde{\rho}_{j_0} \leqslant \hat{\rho}_{j_0}, \qquad \tilde{\rho}_\nu \leqslant \hat{\rho}_\nu \end{aligned} \quad (13.10)$$

Choose $\tilde{\rho} = \tilde{\rho}_{j_0} \prod_{j=1}^{l} \rho_j$. Then (13.10) gives (13.7), so we have the estimate (13.6) for the general term in the last member of (13.5). We also have (13.9) and hence (13.8), so under the assumption (13.10), we get the estimate $\mathscr{O}(1) \prod_{j \in B} |t_j|_{\infty, \rho_j}$ for the quantity (13.5). We have then showed (I_{k+1}). ∎

Now recall that $\langle x_{j_1}, ..., x_{j_k} \rangle_k$ is a finite sum of terms of the form

$$h^{k-l} \langle v_{\mathscr{J}_1} v_{\mathscr{J}_2} \cdots v_{\mathscr{J}_l} \rangle$$

with

$$\mathscr{J}_1 \cup \cdots \cup \mathscr{J}_l = \{1, ..., k\}, \qquad \mathscr{J}_\nu \cap \mathscr{J}_\mu = \varnothing \quad \text{if} \quad \nu \neq \mu, \qquad \# \mathscr{J}_\nu \geqslant 2$$

Let

$$(\tilde{\rho}_{j_1}, ..., \tilde{\rho}_{j_k}) \in \mathscr{R}_k, \qquad \tilde{\rho}_{j_\nu} \leqslant \hat{\rho}_{j_\nu}$$

Then applying (I_k) for $l=0$, we get

$$v_{\mathscr{J}_\nu}(x) = \mathscr{O}(1) \frac{1}{\inf \prod_{\mu \in \mathscr{J}_\nu} \tilde{\rho}_{j_\mu}}$$

and consequently

$$v_{\mathscr{J}_1} v_{\mathscr{J}_2} \cdots v_{\mathscr{J}_l} = \mathscr{O}(1) \prod_{\nu=1}^{l} \frac{1}{\inf \prod_{\mu \in \mathscr{J}_\nu} \tilde{\rho}_{j_\mu}}$$

Hence (for $h \leqslant 1$)

$$\langle x_{j_1}, ..., x_{j_k} \rangle_k = \mathscr{O}(h^{k-[k/2]}) \times \Big[1 \Big/ \Big(\inf_{\substack{l \geqslant 1 \\ \mathscr{J}_1 \cup \cdots \cup \mathscr{J}_l = \{1, ..., k\} \\ \text{(disjoint union)}}} \sup_{\substack{(\tilde{\rho}_{j_1}, ..., \tilde{\rho}_{j_k}) \in \mathscr{R}_k \\ \tilde{\rho}_{j_\mu} \leqslant \hat{\rho}_{j_\mu}}} \prod_{\nu=1}^{l} \inf \prod_{\mu \in \mathscr{J}_\nu} \tilde{\rho}_{j_\mu} \Big) \Big] \quad (13.11)$$

Example 13.1. Let d be a distance on $\{1,..., m\}$ and let

$$\mathscr{R} = \left\{\rho\colon \{1,..., m\} \to]0, +\infty[;\ \frac{\rho(\mu)}{\rho(\nu)} \leqslant \exp d(\mu, \nu)\right\}$$

Put $\hat{\rho}_j(k) = \exp d(j, k)$ and consider

$$\langle x_{j_1}, x_{j_2}\rangle_2 = \mathcal{O}(1)\frac{1}{\sup_{(\tilde{\rho}_{j_1}, \tilde{\rho}_{j_2}) \in \mathscr{R}_2,\ \tilde{\rho}_{j_\nu} \leqslant \hat{\rho}_{j_\nu}} \inf \tilde{\rho}_{j_1}\tilde{\rho}_{j_2}}$$

Choose

$$\tilde{\rho}_{j_1} = \exp d(j_1, \cdot), \qquad \tilde{\rho}_{j_2} = \exp[d(j_1, j_2) - d(j_1, \cdot)]$$

Then we get

$$\langle x_{j_1}, x_{j_2}\rangle_2 = \mathcal{O}(h) \exp -d(j_1, j_2)$$

Example 13.2. We make the same assumptions as in the preceding example, but we shall consider the correlation of arbitrary order $k \geqslant 2$. As before, we put $\hat{\rho}_j(\nu) = \exp d(j, \nu)$ and consider $\langle x_{j_1},..., x_{j_k}\rangle$. Put $\tilde{\rho}_j(\nu) = \exp(1/k)\, d(j, \nu)$, so that

$$(\tilde{\rho}_j(1),..., \tilde{\rho}_j(k)) \in \mathscr{R}_k, \qquad \tilde{\rho}_j(l) \leqslant \hat{\rho}_j(l), \qquad \forall 1 \leqslant l \leqslant k$$

Assume

$$\{j_1,..., j_k\} = \mathscr{J}_1 \cup \cdots \cup \mathscr{J}_l$$

(disjoint union) with $\#\mathscr{J}_j \geqslant 2$.

Fix a $\mathscr{J} = \mathscr{J}_j$ and consider

$$\ln \prod_{n \in \mathscr{J}} \tilde{\rho}_n = \frac{1}{k} \sum_{n \in \mathscr{J}} d(n, \cdot)$$

We notice that

$$\sum \sum_{\substack{n, n' \in \mathscr{J} \\ n \neq n'}} d(n, \nu) + d(\nu, n') = 2 \sum \sum_{\substack{n, n' \in \mathscr{J} \\ n \neq n'}} d(n, \nu) \geqslant 2[(\# J) - 1] \sum_{n \in \mathscr{J}} d(n, \nu)$$

so that

$$\sum_{n \in \mathscr{J}} d(n, \nu) = \frac{1}{2[(\# J) - 1]} \sum \sum_{\substack{n, n' \in \mathscr{J} \\ n \neq n'}} d(n, \nu) + d(\nu, n')$$

$$\geqslant \sum \sum_{\substack{n, n' \in \mathscr{J} \\ n \neq n'}} d(n, n')$$

$$\geqslant \frac{1}{2} \sum_{n \in \mathscr{J}} d(n, \mathscr{J} \setminus \{n\}) \geqslant \frac{1}{2} \sum_{n \in \mathscr{J}} d(n, \mathscr{K} \setminus \{n\})$$

where we put $\mathscr{K} = \{j_1, ..., j_k\}$. It follows that

$$\prod_{j=1}^{l} \inf \prod_{n \in \mathscr{J}_j} \tilde{\rho}_n \geqslant \exp \frac{1}{2k} \sum_{n \in \mathscr{J}} d(n, \mathscr{K} \setminus \{n\})$$

Using this in (13.11), we get

$$\langle x_{j_1}, ..., x_{j_k} \rangle = \mathcal{O}_k(h^{k - [k/2]}) \exp\left[-\frac{1}{2k} \sum_{p=1}^{k} \min_{q \neq p} d(j_p, j_q) \right] \tag{13.12}$$

Example 13.3. We keep the same assumptions as in the preceding examples, but we assume in addition that we are in the one-dimensional case in the sense that

$$1 \leqslant j \leqslant l \leqslant n \leqslant m \Rightarrow d(j, n) = d(j, l) + d(l, n)$$

We restrict attention to the case $k = 3$. Then the only decomposition of $\mathscr{K} = \{j_1, j_2, j_3\}$ is $\mathscr{K} = \mathscr{J}$. We may assume for simplicity that $j_1 < j_2 < j_3$. Define $\tilde{\rho}_{j_1}$, $\tilde{\rho}_{j_2}$, $\tilde{\rho}_{j_3}$ by

	$v \leqslant j_1$	$j_1 \leqslant v \leqslant j_2$	$j_2 \leqslant v \leqslant j_3$	$j_3 \leqslant v$
$\ln \tilde{\rho}_{j_1}$	$d(v, j_1)$	$d(v, j_1)$	$d(j_1, j_2)$	$d(j_1, j_2)$
$\ln \tilde{\rho}_{j_2}$	$d(j_1, j_2)$	$d(v, j_2)$	$d(v, j_2)$	$d(j_2, j_3)$
$\ln \tilde{\rho}_{j_3}$	$d(j_2, j_3)$	$d(j_2, j_3)$	$d(v, j_3)$	$d(v, j_3)$

It is then clear that $(\tilde{\rho}_{j_1}, \tilde{\rho}_{j_2}, \tilde{\rho}_{j_3}) \in \mathscr{R}_3$ and that $\tilde{\rho}_{j_1} \cdot \tilde{\rho}_{j_2} \cdot \tilde{\rho}_{j_3} \geqslant \exp d(j_1, j_3)$. We conclude that

$$\langle x_{j_1}, x_{j_2}, x_{j_3} \rangle = \mathcal{O}(h^2) \exp[-d(j_1, j_3)]$$

PART III

14. APPLICATION TO THE SCHRÖDINGER OPERATOR

14.1. On the Logconcavity of the First Eigenfunction

We first recall that the strict convexity of the potential V implies the same property for $\Phi = -\ln u_1^{(m)}(x)$.[3] We also recall that Proposition 2.1 in ref. 27 gives that if

$$\sup_x \|\text{Hess } V(x) - I\|_{\mathscr{L}(l^2)} = \theta < 1 \tag{14.1}$$

then we have

$$\sup_x \|\text{Hess}\, \Phi(x) - I\|_{\mathscr{L}(l^2)} = \theta/[1+(1-\theta)^{1/2}] \tag{14.2}$$

In order to apply the results of the other parts, we need the more precise:

Proposition 14.1. If $V^{(m)} = x^2/2 + W^{(m)}$ satisfies the assumptions of Theorem 1.6 with $V^{(m)}$ replacing $\Phi^{(m)}$, then the logarithm $-\ln(u_1^{(m)})$ of the first normalized eigenfunction $u_1^{(m)}$ of the Schrödinger operator on $\mathbb{R}^m$ $-\Delta + V^{(m)}$ satisfies the same conditions with $\Phi^{(m)} = -\ln(u_1^{(m)})$ and for some new constants (independent of m).

We observe that Proposition 14.1 is a consequence of Theorem 2.1(A) in ref. 27 for (1.23), (1.24), (1.26), (1.28), and (6.25), of Theorem 2.1(B) in ref. 27 for (1.30) and (1.31), and of Section 4 of the same article to obtain the missing (6.26) and (6.27). The condition of (1.22) is clearly satisfied because we are considering the first eigenvector. The condition (1.21) is not satisfied, but Proposition 6.2 gives what is needed to work without this assumption. We can also remark that this condition is satisfied if, for example, $V^{(m)}$ is an even function [with respect to $x \to -x$. It could also be interesting to analyze the condition (1.9). This condition is deduced from (2.18) in ref. 27 (with $V_0 = x^2/2$ and $V_1 = V$] and the assumption (1.9) on $V_1 - V_0$. The condition (1.8) is not satisfied, but its introduction is not necessary (see, however, Section 6.2).

14.2. FKG Inequalities Relative to Schrödinger

We have observed how the techniques using the maximum principle can be used in order to analyze the correlations. We have recalled in the previous subsection how the assumptions on the potential V of the Schrödinger operator can be transferred to $-\ln u_1$ where $u_1 = u_1^{(m)}$ is the first normalized eigenfunction. We shall see here how the properties on the sign of the correlations can be followed in the same spirit. For this, we come back to Eq. (2.5) in ref. 27. We write

$$\nabla\Phi\nabla(\text{Hess}\,\Phi) + (\text{Hess}\,\Phi)\circ(\text{Hess}\,\Phi) = \text{Hess}\,V + \Delta(\text{Hess}\,\Phi) \tag{14.3}$$

We want to prove that under suitable assumptions on Hess V we obtain the assumptions of Section 12 for Hess Φ. We know already from ref. 27 that the strict convexity of Hess V gives the same property for Hess Φ. As in Section 12, we decompose $P = \text{Hess}\,\Phi$ in the form

$$P = D + P^+ - P^- \tag{14.4}$$

where

$$P_{ij}^{+}(x) = \sup(0, \Phi_{ij}^{(2)}(x)), \qquad \text{for} \quad i \neq j, \qquad P_{ii}^{+}(x) = 0$$

$$P_{ij}^{-}(x) = -\inf(0, \Phi_{ij}^{(2)}(x)), \qquad \text{for} \quad i \neq j, \qquad P_{ii}^{-}(x) = 0$$

and we want to prove that $P^{+}(x) = 0$.

This decomposition is an orthogonal decomposition for the Hilbert–Schmidt scalar product and we assume the following decomposition for $Q = \text{Hess } V$:

$$Q = \tilde{D} - Q^{-} \tag{14.5}$$

We observe that this class of matrices has very nice properties, as described, e.g., in the Appendix in ref. 10. It is then convenient to use the Hilbert–Schmidt norm as in Section 3.1, but we are working now with symmetric matrices. Let x_0 be a point where $\|P^{+}(x)\|$ is maximal. We take the scalar product with $P^{+}(x_0)$ in (14.3). We then observe that the function

$$x \to \text{tr}(\text{Hess } \Phi \circ P^{+}(x_0)) = \text{tr}(P^{+}(x) \circ P^{+}(x_0)) - \text{tr}(P^{-}(x) \circ P^{+}(x_0))$$

attains its maximum at the point x_0 and we get at this point

$$\langle (\text{Hess } \Phi) \circ (\text{Hess } \Phi), P^{+} \rangle \leqslant \langle \text{Hess } V, P^{+} \rangle$$

The r.h.s. is negative and we get finally at the point x_0

$$\text{tr}((\text{Hess } \Phi) \circ (\text{Hess } \Phi) \circ P^{+}) \leqslant 0$$

We shall now compute

$$\begin{aligned}
&\text{tr}(\text{Hess } \Phi) \circ ((\text{Hess } \Phi) - P^{+}) \circ P^{+}) \\
&\quad = \text{tr}((D + P^{+} - P^{-}) \circ (D - P^{-}) \circ P^{+}) \\
&\quad = \text{tr}((P^{+} - P^{-}) \circ (D - P^{-}) \circ P^{+}) \\
&\quad = \text{tr}(D \circ P^{+} \circ P^{+}) + \text{tr}((P^{+} - P^{-}) \circ (-P^{-}) \circ P^{+}) \\
&\quad = \text{tr}((D - P^{-}) \circ P^{+} \circ P^{+}) + \text{tr}(P^{-} \circ P^{-} \circ P^{+})
\end{aligned}$$

From these computations we obtain at the point x_0

$$\text{tr}(P^{+} \circ [D - P^{-} + (\text{Hess } \Phi)] \circ P^{+}) \leqslant 0$$

$$2 \, \text{tr}(P^{+} \circ (\text{Hess } \Phi) \circ P^{+}) \leqslant \text{tr}(P^{+} \circ P^{+} \circ P^{+})$$

Using the assumption of strict convexity, we get the following inequality at the point x_0 [if $\|P^{+}(x_0)\| \neq 0$]:

$$0 < 2\rho \leqslant \|P^{+}(x_0)\|_{\mathscr{L}(l^2)}$$

where ρ is the infimum over x of the smallest eigenvalue of Hess Φ. But if we deform V on $x^2/2$ and keep the assumption of strict convexity, we observe that we have necessarily $P^+(x_0)=0$.

We have proved the following:

Proposition 14.2. If

$$0 < r_0 \leqslant \inf_x \text{Hess } V(x) \tag{14.6}$$

and if Hess $V(x)$ is uniformly bounded and admits the decomposition (14.5), then Hess $\Phi(x)$ has the same property.

Remark 14.3. As in all the proofs we have developed using the maximum principle, we have to justify the argument by a cutoff procedure which is here exactly the same as in ref. 27 (more precisely in Section 2, before the proof of Theorem 2.1). We have first to replace the potential V by a family of potentials

$$V_\varepsilon = \frac{x^2}{2} + \chi_\varepsilon(x)\left(V - \frac{x^2}{2}\right)$$

where $\chi_\varepsilon(x)$ is defined as in Section 5.

14.3. Correlation and Magnetization in the Schrödinger Case

According to Proposition 14.1, we get that Proposition 1.7, Theorem 1.5, and Theorem 1.8 are satisfied if we take $\Phi^{(m)} = -\ln(u_1^{(m)})$.

In the case of Theorem 1.8, we are looking more precisely at the Schrödinger operator relative to the potential

$$V^{(m)}(x, B) = \tfrac{1}{2} \sum_l (x_l - B)^2 + V^{(m)}(x)$$

We have recalled in ref. 13 how this type of potential appears in the context of statistical mechanics (starting with the spin models on two-dimensional lattices introduced by Kac,[18] but adding a magnetic field). We just recall here that the magnetization, which is defined by [see (7.18)] $\langle (x_1 - B) \rangle_m$, can also be seen, using the Feynman–Heilmann formula, as $(\partial\lambda^{(m)}/\partial B)(B)/m$, where $\lambda^{(m)}(B)$ is the first eigenvalue.

ACKNOWLEDGMENT

This article was completed during different stays of the authors at the Mittag-Leffler Institute.

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